ARITHMETICAL PROPERTIES OF A SEQUENCE ARISING FROM AN ARCTANGENT SUM

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Abstract. The sequence $\{x_n\}$ defined by $x_n = (n+x_{n-1})/(1-nx_{n-1})$, with $x_1 = 1$, appeared in the context of some arctangent sums. We establish the fact that $x_n \neq 0$ for $n \geq 4$ and conjecture that x_n is not an integer for $n \geq 5$. This conjecture is given a combinatorial interpretation in terms of Stirling numbers via the elementary symmetric functions. The problem features linkage with a well-known conjecture on the existence of infinitely many primes $1+n^2$, as well as our conjecture that $(1+1^2)(1+2^2)\cdots(1+n^2)$ is not a square for $n > 3$. We present an algorithm that verifies the latter for $n \leq 10^{3200}$.

1. Introduction

The evaluation of arctangent sums of the form

$$\sum_{k=1}^{\infty} \tan^{-1} h(k) \tag{1.1}$$

for a rational function h, appears in the literature from time to time. Throughout the paper $\tan^{-1}(\cdot)$ is defined by its principal branch. In joint work with G. Boros, the third author presented in [3] a systematic study of these sums. There, the reader will find the elementary evaluation

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{2}{k^2} = \frac{3\pi}{4}, \tag{1.2}$$

as well as the more advanced

$$\sum_{k=1}^{\infty} 2^{-k} \tan^{-1} \left(\frac{\sinh 2^k x}{\sin 2^k x} \right) = \tan^{-1} \left(\frac{\tanh x}{\tan x} \right). \tag{1.3}$$

As part of this study, the authors of [3] considered the sequence

$$x_n := \tan \sum_{k=1}^{n} \tan^{-1} k, \quad n \geq 1. \tag{1.4}$$

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The addition formula for $\tan x$ yields the Ricatti-type equation

$$x_n = \frac{x_{n-1}+n}{1-nx_{n-1}}, \tag{1.5}$$

with the initial condition $x_1 = 1$. We prove that $1 - nx_{n-1} \neq 0$ for $n > 1$, so that x_n is well defined. Naturally, $x_n \in \mathbb{Q}$ and the first few values are

$$\left\{ 1,\, -3,\, 0,\, 4,\, -\frac{9}{19},\, \frac{105}{73},\, -\frac{308}{331},\, \frac{36}{43} \right\}. \tag{1.6}$$

Moreover, running (1.5) backwards, we find that $x_0 = 0$. In this paper we settle the conjecture proposed in [3] to the effect that $x_n \neq 0$ for $n \geq 4$. This proof is based on the analysis of the 2-adic valuation of x_n.

Definition 1.1. *Given a prime p and an integer $x \neq 0$, write $x = p^m y$, with y not divisible by p. The exponent m is the p-adic valuation of x, denoted by $m = \nu_p(x)$. This definition is extended to $x = a/b \in \mathbb{Q}$ via $\nu_p(x) = \nu_p(a) - \nu_p(b)$. We leave the value $\nu_p(0)$ as undefined.*

In Section 2 we provide an explicit expression for $\nu_2(x_n)$. This is used to prove that $x_n \neq 0$ for $n \neq 4$. The study of arithmetical properties of the sequence $\{x_n\}$ lead us to propose:

Conjecture 1.2. *For $n \geq 5$, the value x_n is not an integer.*

During the process of developing tables of values for $\ln \Gamma(x+iy)$, J. Todd [16] declared a positive integer m to be *reducible* if there is an identity of the form

$$\tan^{-1} m = \sum f_r \tan^{-1} n_r, \tag{1.7}$$

for some integers f_r, n_r. For example, 13 is reducible since

$$\tan^{-1} 13 = 5\tan^{-1} 1 - \tan^{-1} 2 - \tan^{-1} 4. \tag{1.8}$$

The reducibility of m was characterized in terms of arithmetical properties of m.

Theorem 1.3. *Let $m \in \mathbb{N}$. Then m is reducible if and only if all prime factors of $1+m^2$ occur among the prime factors of $1+k^2$ for $1 \leq k \leq m-1$.*

Theorem 1.4. *Let $m \in \mathbb{N}$. Then m is reducible if and only if the largest prime factor of $1+m^2$ is less than $2m$.*

The question of whether x_n in (1.5) is an integer m corresponds to asking for a reduction of m of a specific type: all f_r must be $+1$ and the integers n_r must be the segment $\{1, 2, \cdots, n\}$.

Some partial results for the resolution of Conjecture 1.2 are given in Section 4. We prove that the sequence $\{x_n : n \geq 5\}$ does not contain two consecutive elements which are integers. In this section we also explore arithmetical conditions on the element x_{n-1}, written in irreducible form as

u/v, in order to obtain $x_n \in \mathbb{Z}$. Proposition 4.3 shows that $x_n \in \mathbb{Z}$ is equivalent to $v - nu$ dividing $1+n^2$. In particular, we show that if $|x_n| \leq n$ and $1+n^2$ is prime, then $x_n \notin \mathbb{Z}$. Note that the existence of infinitely many primes of the form $1+n^2$ is a well-known open problem in Number Theory. Denote by $\mathbb{P}$ the set of prime numbers and introduce

$$\pi_2(n) := \#\{1 \leq k \leq n : 1+k^2 \in \mathbb{P}\}. \tag{1.9}$$

It is conjectured that

$$\pi_2(n) \sim 2C_{quad}\frac{\sqrt{n}}{\ln n}, \tag{1.10}$$

where

$$C_{quad} = \frac{1}{2}\prod_{p\geq 2}\left(1 - \frac{(-1)^{(p-1)/2}}{p-1}\right). \tag{1.11}$$

The expression

$$C_{quad} = \frac{3\zeta(6)}{4G\zeta(3)}\prod_{p\equiv 1 \bmod 4}\left(1+\frac{2}{p^3-1}\right)\left(1-\frac{2}{p(p-1)^2}\right) \tag{1.12}$$

gives an expression for C_{quad} in terms of primes congruent to 1 modulo 4. This is a result of D. Shanks [15]. Here G is the Catalan constant

$$G = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}. \tag{1.13}$$

Theorem 7.10 shows that the condition $|x_n| \leq n$ is valid almost all the time. Thus, for almost all primes of the form $1+n^2$, we conclude that $x_n \notin \mathbb{Z}$.

Section 3 describes a relation between the sequence $\{x_n : n \in \mathbb{N}\}$ and the alternating sums $S_{\pm}(n)$ (see definitions in Section 3) of Newton's elementary symmetric functions,

$$S_k(n) = \sum_{1\leq i_1 < \cdots < i_k \leq n} i_1 \cdots i_k, \quad 1 \leq k \leq n, \tag{1.14}$$

of the numbers $\{1, 2, \cdots, n\}$. Theorem 3.6 states that

$$x_n = \frac{S_-(n)}{S_+(n)}. \tag{1.15}$$

This section also contains explicit analytic expressions for the 2-adic valuations of $S_{\pm}(n)$. In particular it is shown that $\nu_2(S_{\pm}(n)) \geq \lfloor \frac{n+1}{4} \rfloor$.

The point in $\mathbb{Z}^2$ given by

$$\rho(n) := (S_+(n), S_-(n)), \tag{1.16}$$

has an angle equal to

$$\tan^{-1} x_n = \sum_{k=1}^{n} \tan^{-1} k. \tag{1.17}$$

The square of the modulus is given by

$$\omega_n := |\rho(n)|^2 = (1+1^2)(1+2^2)(1+3^2)\cdots(1+n^2). \tag{1.18}$$

We also consider a diophantine equation related to ω_n. In the literature, the solution to

$$1^2 + 2^2 + \cdots + n^2 = m^2 \tag{1.19}$$

is known as *Lucas's square pyramid problem*. The only solutions are $(n, m) \in \{(1,1)\,(24,70)\}$. See [1] and [4] for details. Write

$$R_n(t) = (1+t^2)(1+4t^2)(1+9t^2)\cdots(1+n^2t^2), \tag{1.20}$$

then Lucas' problem amounts to asking whether the coefficient of t^2 in $R_n(t)$ is itself a square.

It is natural that one should investigate the remaining coefficients of R_n, to check whether these are perfect squares. The problem discussed in the present article deals with $\omega_n = R_n(1)$ which is the total sum of the coefficients of $R_n(t)$. Based on extensive numerical evidence, we propose that

Conjecture 1.5. *For $n \geq 4$, the value ω_n is not a square.*

The two conjectures presented above are related. Theorem 5.5 shows that failure of Conjecture 1.5 implies Conjecture 1.2. In Section 5, we consider the product ω_n modulo certain primes. This is used to establish Conjecture 1.5 for n in certain arithmetical progressions, for example, for $n \equiv 1 \bmod 3$. We also describe a sieve that is used to verify this conjecture up to $n \leq 10^{3200}$, in an efficient way. The algorithm is based on the simple observation that, if there is a prime p for which $\nu_p(\omega_n)$ is an odd number, then ω_n is not a square.

Section 6 explores the p-adic properties of ω_n. An explicit 2-adic valuation produces a proof of Conjecture 1.5 for $n \equiv 1, 2 \bmod 4$. This section also discusses the case p odd. Theorem 6.5 states that

$$\nu_p(\omega_n) \sim \frac{2n}{p-1}, \text{ as } n \to \infty. \tag{1.21}$$

The proof of Theorem 6.5 makes use of the solutions to the congruence

$$1 + x^2 \equiv 0 \bmod p^i. \tag{1.22}$$

In the base case $i = 1$, the congruence $1 + x^2 \equiv 0 \bmod p$ has two solutions $\alpha_p \leq \alpha_p^*$ in the interval $1 \leq x \leq p-1$. The first root α_p satisfies

$$\sqrt{p-1} \leq \alpha_p \leq (p-1)/2. \tag{1.23}$$

These two roots produce solutions to the congruences modulo p^i. For example, for modulus p^2, we have that $1 + x^2 \equiv 0 \bmod p^2$. Therefore, $x = \alpha + tp$

for some $t \in \{0, 1, \cdots, p-1\}$ (or $x = \alpha^* + tp$). The bounds on α_p shows that $1 + \alpha_p^2 = pb_1$, with $b_1 \not\equiv 0 \bmod p$. The congruence $1 + x^2 \equiv 0 \bmod p^2$ yields $2\alpha_p t \equiv -b_1 \bmod p$ and t is uniquely determined, say $t = t_1$. We let

$$\alpha_{p^2} := \alpha_p + t_1 p. \tag{1.24}$$

This argument produces a double sequence of numbers

$$\alpha_p,\, \alpha_{p^2},\, \alpha_{p^3}, \cdots \quad \text{and} \quad \alpha_p^*,\, \alpha_{p^2}^*,\, \alpha_{p^3}^*, \cdots \tag{1.25}$$

such that

$$1 + x^2 \equiv 0 \bmod p^i \text{ if and only if } x \equiv \alpha_{p^i} \text{ or } x \equiv \alpha_{p^i}^* \bmod p^i. \tag{1.26}$$

The construction shows that

$$\alpha_{p^i} \equiv \alpha_{p^{i-1}} \bmod p^{i-1}. \tag{1.27}$$

Section 5 presents a connection between Conjecture 1.5 and primes of the form $1+x^2$. We show that the existence of an integer x in the range $[\sqrt{n},\, n]$, such that $1 + x^2$ is a prime, implies Conjecture 1.5.

The question of whether x_n is an integer suggests the study of the sequence of fractional parts defined by

$$y_n := \{x_n\} = x_n - \lfloor x_n \rfloor.$$

Figure 1 shows the sequence $\{x_n\}$ for $1 \leq n \leq 100000$, and Figure 2 shows the corresponding fractional parts. Observe the presence of *granular* regions combined with some *solid curve* regions. This combination persists as n increases.

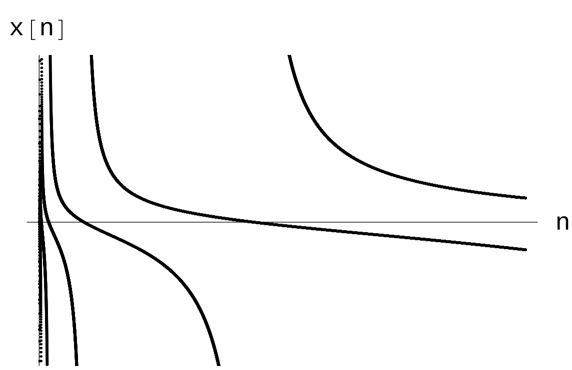

FIGURE 1. The sequence x_n

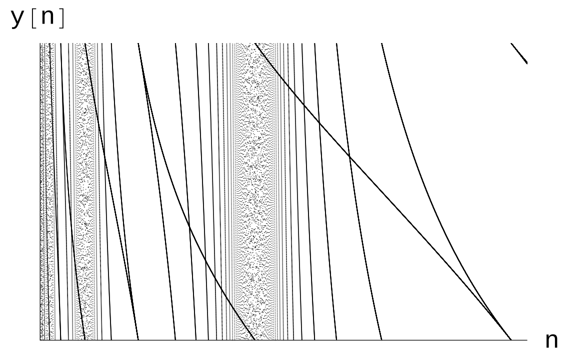

FIGURE 2. The fractional part of x_n

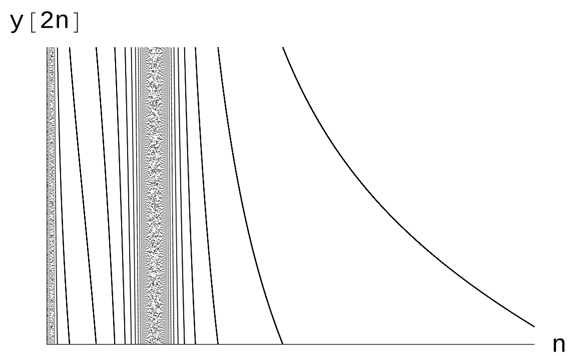

FIGURE 3. The fractional part of the sequence x_{2n}

The sequence $\{y_n\}$ has many interesting dynamical properties. For instance, we point out the *lack of intrusion* between the curves and the granular region observed in Figure 3. These phenomena will be considered in future work.

The last section contains miscellaneous topics and future directions inspired by the results presented in this paper.

2. The 2-adic valuations of x_n

Let $m \in \mathbb{Z}$ and p a prime number. This section begins the discussion of the properties of the p-adic valuation of x_n, defined in (1.1). The explicit evaluation of $\nu_2(x_n)$ is used to establish that $x_n \neq 0$ for $n \geq 4$ by showing that $\nu_2(x_n)$ is well defined.

Theorem 2.1. *Let $n \geq 4$. Then $x_n \neq 0$.*

The proof of the theorem is based on the following result.

Theorem 2.2. *Let $n \in \mathbb{N}$ and $N = \lfloor \frac{n}{4} \rfloor$. The 2-adic valuation of x_n is given by*

$$\nu_2(x_n) = \begin{cases} \nu_2(2N(N+1)) & \text{if} \quad n \equiv 0, 3 \bmod 4, \\ 0 & \text{if} \quad n \equiv 1, 2 \bmod 4. \end{cases}$$

The demonstration of this proposition is divided into several steps.

An elementary inductive argument, using (1.5) in the form

$$x_{n+1} = \frac{x_n + n + 1}{1 - (n+1)x_n}, \tag{2.1}$$

gives the first result.

Lemma 2.3. *Let $n, k \in \mathbb{N}$. There exist polynomials P_k and Q_k for which*

$$x_{n+k} = \frac{P_k(n)x_n + Q_k(n)}{P_k(n) - Q_k(n)x_n}. \tag{2.2}$$

The polynomials P_k, Q_k satisfy the recurrences

$$\begin{aligned} P_{k+1}(n) &= P_k(n) - (n+k+1)Q_k(n), \\ Q_{k+1}(n) &= Q_k(n) + (n+k+1)P_k(n), \end{aligned} \tag{2.3}$$

with initial conditions $P_1(n) = 1$ and $Q_1(n) = n + 1$.

We now establish Theorem 2.2 for the case $n \equiv 0 \bmod 4$.

Proposition 2.4. *Let $n \in \mathbb{N}$. Then $\nu_2(x_{4n}) = \nu_2(2n(n+1))$.*

Proof. The proof is divided into cases according to the value of $\nu_2(n)$. Write $n = 2^{\nu_2(n)}t$, with t odd.

Case 1: $\nu_2(n) = 1$. We write $t = 2m + 1$ and we need to prove

$$\nu_2(x_{16m+8}) = 2. \tag{2.4}$$

The proof is by induction starting at

$$\nu_2(x_8) = \nu_2\left(-\tfrac{36}{43}\right) = 2. \tag{2.5}$$

To continue the inductive procedure we need a relation between $x_{16(m+1)+8}$ and x_{16m+8}.

Claim: there are odd integers c_1, c_2 such that

$$x_{16(m+1)+8} = \frac{8c_1 + c_2 x_{16m+8}}{c_2 - 8c_1 x_{16m+8}}. \tag{2.6}$$

Lemma 2.3 gives

$$x_{16(m+1)+8} = \frac{P_{16}(16m+8)x_{16m+8} + Q_{16}(16m+8)}{P_{16}(16m+8) - Q_{16}(16m+8)x_{16m+8}}, \tag{2.7}$$

and the representation (2.6) comes from a direct symbolic calculation:

$$\begin{aligned} P_{16}(16m+8) &= 16 \bmod 32, \\ Q_{16}(16m+8) &= 128 \bmod 256. \end{aligned}$$

From (2.6) we obtain

$$\begin{aligned} \nu_2(x_{16(m+1)+8}) &= \nu_2\left(\frac{8c_1 + c_2 x_{16m+8}}{c_2 - 8c_1 x_{16m+8}}\right) \\ &= \nu_2\left(4 \cdot \frac{2c_1 + c_2\,\frac{1}{4}x_{16m+8}}{c_2 - 8c_1 x_{16m+8}}\right) \end{aligned}$$

and, using the inductive hypothesis $\nu_2(\frac{1}{4}x_{16m+8}) = 0$, we conclude the proof of Case 1.

Case 2 $\nu_2(n) = 0$, or $\nu_2(n) > 1$. The claim to be proven is

$$\nu_2(x_{4n}) = \nu_2(2n(n+1)), \tag{2.8}$$

where $n = 2^{\nu_2(n)}t$ with t odd. Proceed by induction.

Claim: there are odd integers α_1, α_2 such that

$$x_{4n+4} = \frac{\alpha_2 x_{4n} + 4(n+1)\alpha_1}{\alpha_2 - 4(n+1)\alpha_1 x_{4n}}. \tag{2.9}$$

This representation comes from Lemma 2.3 in the form

$$x_{4n+4} = \frac{P_4(4n)x_{4n} + Q_4(4n)}{P_4(4n) - Q_4(4n)x_{4n}}, \tag{2.10}$$

and the observation that $P_4(4n) = 2 \bmod 4$, and $Q_4(4n) = 8 \bmod 16$.

We now consider the 2-adic valuation of (2.9). First of all,

$$\nu_2(\alpha_2 - 4(n+1)\alpha_1 x_{4n}) = 0, \tag{2.11}$$

so that

$$\nu_2(x_{4n+4}) = \nu_2(4(n+1)\alpha_1 + \alpha_2 x_{4n}). \tag{2.12}$$

We now prove by induction that

$$\nu_2(x_{4n+4}) = \nu_2(2(n+1)(n+2)). \tag{2.13}$$

Start with

$$\begin{aligned} \nu_2\left(\frac{x_{4n+4}}{2(n+1)(n+2)}\right) &= \nu_2\left(\frac{2\alpha_1}{n+2} + \frac{\alpha_2 x_{4n}}{2(n+1)(n+2)}\right) \\ &= \nu_2\left(\alpha_1 + \frac{n}{n+2}(-\alpha_1 + \mu\alpha_2)\right), \end{aligned} \tag{2.14}$$

with

$$\mu = \frac{x_{4n}}{2n(n+1)}. \tag{2.15}$$

The inductive hypothesis states that μ is odd. Therefore, $\nu_2(\alpha_2\mu - \alpha_1) \geq 1$.

From $n = 2^{\nu_2(n)}t$, we see that if $\nu_2(n) = 0$ then n is odd and the term in (2.14) is zero. On the other hand, if $\nu_2 > 1$, then

$$\nu_2\left(\frac{n}{n+2}\right) = \nu_2\left(\frac{2^{\nu_2(n)-1}t}{2^{\nu_2(n)-1}t+1}\right) = \nu_2(n) - 1 > 0, \tag{2.16}$$

and the term in (2.14) vanishes again. For either case, the proof of Proposition 2.4 is complete. □

The next step is to establish the result in Theorem 2.2 for the case $n \equiv 3 \bmod 4$.

Proposition 2.5. *Let $n \in \mathbb{N}$. Then $\nu_2(x_{4n+3}) = \nu_2(2n(n+1))$.*

Proof. We have the representation

$$x_{4n+3} = \frac{a_1 + a_2x_{4n}}{a_2 - a_1x_{4n}}, \tag{2.17}$$

with a_1 even and a_2 odd. Indeed, Lemma 2.3 yields

$$x_{4n+3} = \frac{P_3(4n)x_{4n} + Q_3(4n)}{P_3(4n) - Q_3(4n)x_{4n}}, \tag{2.18}$$

and an explicit evaluation of $P_3(4n)$ and $Q_3(4n)$ produces (2.17) with

$$a_1 = 16n(n+1)(2n+1) \text{ and } a_2 = 24n^2 + 24n + 5. \tag{2.19}$$

From Proposition 2.4, we obtain that $\nu_2(x_{4n}) = \nu_2(2n(n+1)) \geq 2$, so that $\nu_2(a_2 - a_1x_{4n}) = 0$. We conclude that $\nu_2(x_{4n+3}) = \nu_2(a_1 + a_2x_{4n})$. Now observe that

$$\begin{aligned} \nu_2\left(\frac{x_{4n+3}}{2n(n+1)}\right) &= \nu_2\left(\frac{a_1}{2n(n+1)} + a_2 \cdot \frac{x_{4n}}{2n(n+1)}\right) \\ &= \nu_2\left(8(2n+1) + a_2 \cdot \frac{x_{4n}}{2n(n+1)}\right) = 0, \end{aligned}$$

because a_2 and $\frac{x_{4n}}{2n(n+1)}$ are odd. The proof of Proposition 2.5 is complete. □

We continue with the proof of Theorem 2.2 for the remaining cases $n \equiv 1, 2 \bmod 4$.

Proposition 2.6. *Let $n \in \mathbb{N}$ and assume $n \equiv 1, 2 \bmod 4$. Then $\nu_2(x_n) = 0$.*

Proof. Let $m = n - 2$, so that $m \equiv 3, 0 \bmod 4$. Lemma 2.3 gives

$$\begin{aligned} x_{m+2} &= \frac{P_2(m+1)x_m + Q_2(m+1)}{P_2(m+1) - Q_2(m+1)x_m} \\ &= \frac{(m+1)(m+2)x_m - x_m - (2m+3)}{(2m+3)x_m + (m+1)(m+2) - 1}. \end{aligned} \tag{2.20}$$

From Propositions 2.4 and 2.5 we have that x_m is even. Then (2.20) shows that x_n is odd, as claimed. □

The proof of Theorem 2.2 is complete. In particular, the expression for $\nu_2(x_n)$ is well defined, showing that $x_n \neq 0$.

Corollary 2.7. *For any $n \in \mathbb{N}$, the value $\nu_2(x_n)$ is well defined and the element x_n is finite. Moreover, $x_n \neq -(n+1),\ 1/(n+1)$.*

Section 7.4 contains information about the p-adic valuation of x_n.

3. A representation by symmetric functions

In this section we consider the elementary symmetric functions of the symbols

$$\mathbb{A}_n := \{\lambda_1,\, \lambda_2, \cdots, \lambda_n\}, \tag{3.1}$$

defined by

$$S_k(\mathbb{A}_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \leq k \leq n. \tag{3.2}$$

As usual $S_0(\mathbb{A}_n) = 1$. The sequence $\{x_n\}$ is now expressed in terms of these symmetric functions for a specific choice of the symbols $\{\lambda_i\}$.

Definition 3.1. *The* even *and* odd *components of the symmetric function $\mathbb{A}_n$ are, respectively,*

$$S_+(\mathbb{A}_n) := \sum_{k \geq 0} (-1)^k S_{2k}(\mathbb{A}_n), \text{ and } S_-(\mathbb{A}_n) := \sum_{k \geq 0} (-1)^k S_{2k+1}(\mathbb{A}_n). \tag{3.3}$$

The next few properties are elementary.

Proposition 3.2. *The generating function of the symmetric functions S_k is given by*

$$G_n(z) = \prod_{j=1}^{n} (1 + z\lambda_j) = \sum_{k=0}^{n} S_k(\mathbb{A}_n) z^k. \tag{3.4}$$

Moreover, the functions S_k satisfy the recurrence relation

$$S_{k+1}(\mathbb{A}_{n+1}) = S_{k+1}(\mathbb{A}_n) + \lambda_{n+1} S_k(\mathbb{A}_n). \tag{3.5}$$

The following result follows directly from (3.5).

Corollary 3.3. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned} \lambda_{n+1} S_+(\mathbb{A}_n) &= S_-(\mathbb{A}_{n+1}) - S_-(\mathbb{A}_n), \\ -\lambda_{n+1} S_-(\mathbb{A}_n) &= S_+(\mathbb{A}_{n+1}) - S_+(\mathbb{A}_n), \\ \lambda_n S_+(\mathbb{A}_{n+1}) &= (\lambda_n + \lambda_{n+1}) S_+(\mathbb{A}_n) - \lambda_{n+1}(\lambda_n^2 + 1) S_+(\mathbb{A}_{n-1}), \\ \lambda_n S_-(\mathbb{A}_{n+1}) &= (\lambda_n + \lambda_{n+1}) S_-(\mathbb{A}_n) - \lambda_{n+1}(\lambda_n^2 + 1) S_-(\mathbb{A}_{n-1}). \end{aligned} \tag{3.6}$$

Corollary 3.4. *Assume* $\lambda_j \neq 0$ *and define* $\mathbb{A}_n^* = \{\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_n^{-1}\}$ *and*

$$\Lambda_n := \prod_{j=1}^{n} \lambda_j. \tag{3.7}$$

Then the parity-dependent identities

$$\begin{aligned} S_+(\mathbb{A}_{2n}) &= (-1)^n \Lambda_n S_+(\mathbb{A}_{2n}^*), \\ S_-(\mathbb{A}_{2n}) &= (-1)^{n-1} \Lambda_n S_-(\mathbb{A}_{2n}^*), \\ S_+(\mathbb{A}_{2n+1}) &= (-1)^n \Lambda_n S_-(\mathbb{A}_{2n+1}^*), \\ S_-(\mathbb{A}_{2n+1}) &= (-1)^n \Lambda_n S_+(\mathbb{A}_{2n+1}^*). \end{aligned} \tag{3.8}$$

hold. It follows that

$$\frac{S_-(\mathbb{A}_{2n})}{S_+(\mathbb{A}_{2n})} = -\frac{S_-(\mathbb{A}_{2n}^*)}{S_+(\mathbb{A}_{2n}^*)}, \quad \frac{S_-(\mathbb{A}_{2n+1})}{S_+(\mathbb{A}_{2n+1})} = \frac{S_+(\mathbb{A}_{2n+1}^*)}{S_-(\mathbb{A}_{2n+1}^*)}.$$

The functions S_+ and S_- in (3.3) can be given a matrix formulation:

Lemma 3.5. *The functions* S_+ *and* S_- *satisfy*

$$\begin{pmatrix} S_+(\mathbb{A}_n) & -S_-(\mathbb{A}_n) \\ S_-(\mathbb{A}_n) & S_+(\mathbb{A}_n) \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} 1 & -\lambda_j \\ \lambda_j & 1 \end{pmatrix} \tag{3.9}$$

Choose the symbols $\lambda_k = k$ for $1 \leq k \leq n$, and for simplicity write $S_k(n)$ instead of $S_k(\mathbb{A}_n)$.

Theorem 3.6. *Assume* $n \geq 0$. *Then*

$$x_n = \frac{S_-(n)}{S_+(n)} \tag{3.10}$$

where

$$S_-(n) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k S_{2k+1}(n), \text{ and } S_+(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k S_{2k}(n), \tag{3.11}$$

are the odd and even parts of $\{S_k(n)\}$ *respectively.*

Proof. The result is established by induction. Corollary 3.3, the recurrence (1.5), and the induction hypothesis yield

$$\begin{aligned} x_{n+1} &= \frac{x_n + (n+1)}{1 - (n+1)x_n} \\ &= \frac{S_-(n) + (n+1)S_+(n)}{S_+(n) - (n+1)S_-(n)}. \end{aligned}$$

This proves the assertion. □

In this case Corollary 3.3 becomes:

Corollary 3.7. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} nS_+(n-1) &= S_-(n) - S_-(n-1) \\ -nS_-(n-1) &= S_+(n) - S_+(n-1). \end{aligned}$$

Moreover

$$nS_{\pm}(n+1) = (2n+1)S_{\pm}(n) - (n+1)(n^2+1)S_{\pm}(n-1). \tag{3.12}$$

The value of the 2-adic valuations of S_+ and S_- are described next.

Theorem 3.8. *The even partial sequences satisfy*

$$\nu_2(S_+(n)) = \lfloor \tfrac{n+1}{4} \rfloor, \tag{3.13}$$

and the odd components satisfy

$$\nu_2(S_-(n)) = \begin{cases} \lfloor \frac{n+1}{4} \rfloor & \text{if } n \equiv 1,\, 2 \bmod 4, \\ \lfloor \frac{n+1}{4} \rfloor + \nu_2\left(2\lfloor \frac{n}{4} \rfloor (\lfloor \frac{n}{4} + 1 \rfloor)\right) & \text{if } n \equiv 0,\, 3 \bmod 4. \end{cases} \tag{3.14}$$

In particular, $\nu_2(S_+(n))$ and $\nu_2(S_-(n))$ are bounded from below by $\lfloor \frac{n+1}{4} \rfloor$.

Proof. The second identity in Corollary 3.3 gives

$$(n+1)S_+(n) = S_-(n+1) - S_-(n), \tag{3.15}$$

and (3.10) yields

$$(x_n + n + 1)S_+(n) = x_{n+1}S_+(n+1). \tag{3.16}$$

This identity is now used to show that

$$\nu_2(S_+(4m-1)) = \nu_2(S_+(4m)) = \nu_2(S_+(4m+1)) = \nu_2(S_+(4m+2)). \tag{3.17}$$

First let $n = 4m$ in (3.16) to produce

$$(x_{4m} + 4m + 1)S_+(4m) = x_{4m+1}S_+(4m+1). \tag{3.18}$$

Theorem 2.2 shows that x_{4m} is even and x_{4m+1} is odd, therefore

$$\nu_2(S_+(4m)) = \nu_2(S_+(4m+1)). \tag{3.19}$$

Then put $n = 4m+1$ in (3.16) to obtain

$$(x_{4m+1} + 4m + 2)S_+(4m+1) = x_{4m+2}S_+(4m+2). \tag{3.20}$$

Theorem 2.2 shows that x_{4m+1} and x_{4m+2} are odd, so that

$$\nu_2(S_+(4m+1)) = \nu_2(S_+(4m+2)). \tag{3.21}$$

The final step in the proof of (3.17) comes from the second formula in Corollary 3.3 and (3.10) which yields

$$S_+(n+1) = (1 - (n+1)x_n)S_+(n). \tag{3.22}$$

Now replace $n = 4m - 1$ to obtain

$$S_+(4m) = (1 - 4m \cdot x_{4m-1})S_+(4m-1). \tag{3.23}$$

This implies $\nu_2(S_+(4m)) = \nu_2(S_+(4m-1))$.

The evaluation

$$\nu_2(S_+(4m)) = m, \tag{3.24}$$

is now established by induction. The periodicity of $\nu_2(S_+)$ then produces (3.13). The value $S_+(1) = -10$ gives $\nu_2(S_+(1)) = 1$ and (3.17) shows that (3.24) is correct for $m = 1$. The inductive step is achieved by putting $n = 4m + 2$ in (3.22) to obtain

$$S_+(4m+3) = (1 - (4m+3)x_{4m+2})S_+(4m+2). \tag{3.25}$$

Assume for the moment that

$$\nu_2(1 - (4m+3)x_{4m+2}) = 1, \tag{3.26}$$

and use (3.25) to obtain

$$\nu_2(S_+(4m+3)) = 1 + \nu_2(S_+(4m+2)). \tag{3.27}$$

The induction hypothesis and (3.17) complete the proof of (3.24).

To prove (3.26) use (2.20) with $n = 4m$ to obtain

$$x_{4m+2} = \frac{(4m+1)(4m+2)x_{4m} - x_{4m} - (8m+3)}{(8m+3)x_{4m} + (4m+1)(4m+2) - 1}. \tag{3.28}$$

This can be expressed as

$$[(8m+3)x_{4m} + t_m][1 - (4m+3)x_{4m+2}] = 2[u_m - v_m x_{4m}], \tag{3.29}$$

with $u_m = 24m^2 + 24m + 5$, $v_m = 16m(1+m)(2m+1)$ and $t_m = 2(4m+1)(2m+1) - 1$. Thus u_m and t_m are odd and v_m is even. Theorem 2.2 shows that x_{4m} is even, so the 2-adic valuation of the right hand side of (3.29) is 1. On the left hand side of (3.29), the first term is odd, so (3.26) must hold. The proof of (3.13) is complete. The expression (3.14) follows directly from (3.10). □

4. Conditions for integrality of the sequence $\{x_n\}$

The next goal of this paper is to examine the possibility that x_n is an integer for $n \geq 5$. Recall that the first terms of this sequence are $\{0,\, 1,\, -3, 0,\, 4,\, -9/19\,\}$.

Theorem 4.1. *Let $n > 4$. Then, x_{n-1} and x_n cannot both be integers.*

Proof. Assume

$$x_n = \frac{n + x_{n-1}}{1 - n x_{n-1}}, \tag{4.1}$$

and that $x_{n-1}, x_n \in \mathbb{Z}$. Then $|x_n| \geq 1$ because it has been established that $x_n \neq 0$ for $n \neq 3$. Therefore

$$|n + x_{n-1}| \geq |1 - n x_{n-1}|. \tag{4.2}$$

The discussion of this inequality is divided into four cases according to the sign of the expressions in (4.2).

Case 1: $n + x_{n-1} \geq 0$ and $1 - nx_{n-1} \geq 0$. This is equivalent to $-n \leq x_{n-1} \leq \frac{1}{n}$. The fact that $x_{n-1} \neq 0$ produces $-n \leq x_{n-1} \leq -1$. In this case (4.2) becomes $(1+n)x_{n-1} \geq 1-n$. Let $t_{n-1} = -x_{n-1}$, so that $1 \leq t_{n-1} \leq n$. Then $(1+n)t_{n-1} \leq n-1$ and this contradicts $t_{n-1} \geq 1$.

Case 2: $n + x_{n-1} \geq 0$ and $1 - nx_{n-1} \leq 0$. This is equivalent to $x_{n-1} \geq 0$. Then (4.2) becomes $n + x_{n-1} \geq nx_{n-1} - 1$, that yields $x_{n-1} \leq \frac{n+1}{n-1}$. For $n > 3$, this implies $x_{n-1} < 2$, that is, $x_{n-1} = 1$. Thus,

$$x_n = \frac{n+1}{1-n}, \tag{4.3}$$

and it follows that $x_n < 0$. Moreover, for $n > 3$,

$$x_n = -\frac{n+1}{n-1} > -2. \tag{4.4}$$

This shows that $x_n = -1$, contradicting (4.3).

Case 3: $n + x_{n-1} \leq 0$ and $1 - nx_{n-1} \geq 0$. This is equivalent to $x_{n-1} \leq -n$. In this case (4.2) becomes

$$-n - x_{n-1} \geq 1 - nx_{n-1}, \tag{4.5}$$

that is equivalent to

$$x_{n-1} \geq \frac{n+1}{n-1}. \tag{4.6}$$

This contradicts the fact that $x_{n-1} \leq -n$.

Case 4: $n + x_{n-1} \leq 0$ and $1 - nx_{n-1} \leq 0$. This is equivalent to $x_{n-1} \leq -n$ and $x_{n-1} \geq 1/n$. This situation does not occur. □

The more general question of whether it is possible to have integers a, b and c such that

$$\frac{a+b}{1-ab} = c, \tag{4.7}$$

is considered next. All integer solutions to (4.7) are determined. The authors wish to thank B. Scher for suggesting this result.

Theorem 4.2. *The values* $(1, 2, -3)$ *and* $(0, b, b)$, *with* $b \in \mathbb{Z}$ *are solutions to* (4.7). *All other integer solutions are obtained from these by using the fact that, if* (a, b, c) *solves* (4.7), *then so do* $(-a, -b, -c)$, $(a, -c, -b)$, $(c, a, -b)$ *and* $(b, -c, -a)$.

Proof. There are no solutions with $|a|$, $|b|$, $|c| \geq 2$. Indeed, it follows that

$$|a| + |b| \geq |a+b| \geq 2|1-ab| \geq 2(|ab| - 1), \tag{4.8}$$

and this implies that $2|a||b| - 2 \leq |a| + |b|$. Thus, $|a| + 2 \geq (2|a| - 1)|b| \geq 4|a| - 2$, that is, $3|a| \leq 4$. This is a contradiction.

The solutions $(0,b,b)$, $(a,0,a)$, $(a,-a,0)$ correspond to the trivial case in which one of the variables vanishes. The case $a=1$ yields

$$c = \frac{1+b}{1-b} = -1 - \frac{2}{b-1}, \tag{4.9}$$

and it follows that $b-1 = \pm 1, \pm 2$. This produces the solutions

$$(1,0,1),\ (1,2,-3), (1,-1,0), (1,3,-2). \tag{4.10}$$

A similar analysis can be made with $a=-1$ and also $|b|=1$ and $|c|=1$. The statement about the new solutions admits a direct verification. □

Assumption. Let $n \geq 5$ be an index for which $x_n \in \mathbb{Z}$. Write

$$x_{n-1} = \frac{u}{v} \text{ with } gcd(u,v)=1. \tag{4.11}$$

We now explore some arithmetical properties of $x_{n-1} \in \mathbb{Q}$.

Proposition 4.3. *Let $n \geq 5$. Then $x_n \in \mathbb{Z}$ if and only if $v-nu$ divides $1+n^2$.*

Proof. The result follows from $\gcd(v-nu,u)=1$ and $x_n = n+u(1+n^2)/(v-nu)$. □

Lemma 4.4. *Assume $x_n \in \mathbb{Z}$ and define $c := gcd(x_n-n, 1+nx_n)$. Then c divides $1+n^2$.*

Proof. The recurrence for x_n yields

$$\frac{u}{v} = \frac{x_n - n}{1+nx_n}. \tag{4.12}$$

Therefore $x_n - n = cu$ and $1+nx_n = cv$. The relation $1+n^2 = c(v-nu)$ and Proposition 4.3 show that c must divide $1+n^2$. □

Theorem 4.5. *Let $n \geq 5$. Assume $|x_n| \leq n$ and that $1+n^2$ is prime. Then $x_n \notin \mathbb{Z}$.*

Proof. Suppose $x_n = m \in \mathbb{Z}$. Then (3.10) gives $mS_+(n) = S_-(n)$. Corollary 3.7 yields

$$(m-n)S_+(n-1) = (1+mn)S_-(n-1). \tag{4.13}$$

The identity $1+n^2 = (1+mn)-n(m-n)$, shows that $c = \gcd(m-n, 1+mn)$ divides $1+n^2$. Similarly c divides $1+m^2$. It follows that $c=1$ or $c=1+n^2$. In the latter case, $m=n$, since $|m| \leq n$. This yields $S_-(n-1)=0$. Therefore $x_{n-1}=0$ and this is a contradiction. Therefore $c=1$. The relation (4.13) now gives

$$S_+(n-1) = 1+mn, \text{ and } S_-(n-1) = m-n. \tag{4.14}$$

Theorem 3.8 shows that 2 divides $S_+(n-1)$ and $S_-(n-1)$, contradicting $c=1$. □

Note. The hypothesis $|x_n| \leq n$ in the above theorem holds for almost every $n \in \mathbb{N}$. Theorem 7.10 actually gives a sharper bound $|x_n| \leq \lfloor \frac{n}{2} \rfloor + 1$.

Corollary 4.6. *Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Assume $gcd(m-n, 1+mn) = 1$. Then $x_n \neq m$.*

5. A related diophantine equation

The sequence

$$\omega_n := (1+1^2)(1+2^2)(1+3^2)\cdots(1+n^2), \tag{5.1}$$

that appeared as the modulus of the point $\eta(n) = (S_+(n), S_-(n))$, is studied in this section. Numerical calculations suggest that ω_n is never a square. This is the content of Conjecture 1.5:

The diophantine equation $\omega_n = m^2$ has no solutions for $n \neq 3$.

The arithmetical properties of ω_n investigated in this section deal with ω_n modulo a prime p. Every prime divisor of ω_n must satisfy $p \equiv 1 \bmod 4$, so we consider here $p \equiv 3 \bmod 4$. The next section deals with primes $p \equiv 1 \bmod 4$.

Observe first that the simpler question, whether

$$(n+1)! = (1+1)(1+2)(1+3)(1+4)\cdots(1+n), \tag{5.2}$$

is a square, can be answered in the negative. This is the natural analog of Conjecture 1.5 with an immediate generalization to odd exponents. See Proposition 5.1.

Note. The equation

$$n! + k = m^2 \tag{5.3}$$

was considered by H. Brocard [5, 6] and then, unaware of its history, it was discussed by S. Ramanujan [14], page 327. B. Berndt and W. Galway [2] reported on the equation

$$\left(\frac{n!+1}{p} \right) = 0 \text{ or } 1, \text{ where } p \text{ is a prime.} \tag{5.4}$$

The only solutions of (5.3) or (5.4) are $n = 4, 5, 7$. Here $\left(\frac{a}{p}\right)$ is the Legendre symbol, defined in (5.14).

Proposition 5.1. *The diophantine equation*

$$\Omega_\mu(n) := (1+1^\mu)(1+2^\mu)\cdots(1+n^\mu) = m^2 \tag{5.5}$$

has no solutions for $n \geq 2$ and μ odd.

Proof. Start with the factorization

$$\Omega_\mu(n) = \prod_{j=1}^{n} (1+j) \times \frac{1+j^\mu}{1+j}. \tag{5.6}$$

The case $n = 2$, 3 are checked directly. For $n \geq 4$, Bertrand's postulate [13] gives the existence of a prime p in the range $\lfloor n/2 \rfloor < p < 2\lfloor n/2 \rfloor$. This yields $p < n$ and $2p \geq n+1$, so that p^2 does not divide $(n+1)!$. Thus, $\nu_p((n+1)!) = 1$. This prime cannot divide the term involving the cyclotomic polynomial $(1+j^\mu)/(1+j)$. Therefore $\Omega_\mu(n)$ cannot be a square. □

In sharp contrast to Proposition 5.1, it seems that the problem is more resilient when μ is even. The results described below offer some evidence towards the validity of Conjecture 1.5, when $\mu = 2$.

The symmetric functions S_+ and S_- defined in (3.11) are analyzed next. The first result follows directly from the definitions of G_n in (3.4).

Lemma 5.2. *Let $n \in \mathbb{N}$. Then*

$$G_n(i) = S_+(\mathbb{A}_n) + iS_-(\mathbb{A}_n). \tag{5.7}$$

The modulus of (5.7) gives the Pythagorean relation

$$\prod_{j=1}^{n} (1+\lambda_j^2) = S_+^2(\mathbb{A}_n) + S_-^2(\mathbb{A}_n), \tag{5.8}$$

This, in fact, can be considered as a generalization to Euler's product for sums of two squares:

$$\prod_{j=1}^{2} (1+\lambda_j^2) = (1+\lambda_1\lambda_2)^2 + (\lambda_1 - \lambda_2)^2.$$

Writing $\lambda_1 = a/b$ and $\lambda_2 = c/d$ gives the classical form

$$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2. \tag{5.9}$$

This identity proves that products of numbers representable as sums of two squares are also representable.

The special case $\lambda_j = j$ produces

$$G_n(i) = S_+(n) + iS_-(n), \tag{5.10}$$

and the modulus of this relation yields

$$\omega_n = S_+(n)^2 + S_-(n)^2. \tag{5.11}$$

The following statement is an elementary consequence of the representation (3.10).

Proposition 5.3. *Assume that for $n \geq 5$, the term x_n is an integer m. Then*

$$\omega_n = \prod_{j=1}^{n} (1+j^2) = (1+m^2) S_+^2(n). \tag{5.12}$$

Proof. Immediate from (3.10) and (5.11). □

Corollary 5.4. *If $n \equiv 0, 3 \bmod 4$, then m is even, if $n \equiv 1, 2 \bmod 4$, then m is odd.*

Proposition 5.3 implies that, if $x_n = m$ for some $m \in \mathbb{Z}$, then

$$Y_{n,m} := (1+m^2)\omega_n, \tag{5.13}$$

is a perfect square. This cannot be excluded on general grounds: there are examples for which this happens, for instance,

$$(1+21^2)\omega_5 = (1+21^2)(1+1^2)(1+2^2)(1+3^2)(1+4^2)(1+5^2) = 4420^2.$$

The authors wish to thank James McLaughlin for this example.

The next result gives a sufficient condition for $x_n \notin \mathbb{Z}$.

Theorem 5.5. *Assume that for $n \geq 5$, the term ω_n is a square. Then x_n is not an integer.*

Proof. Proposition 5.3 implies that $Y_{n,m} = (1+m^2)\omega_n$ is a square. If ω_n is also a square, then so is $1+m^2$. This is impossible. □

Note. Interestingly enough, the hypothesis in Theorem 5.5 never holds and this has become the content of Conjecture 1.2, instead. The remainder of the section explores the impossibility that ω_n is a square.

Modular properties. The term ω_n is now considered modulo a fixed prime p. This is used to establish that ω_n is not a square for a specific class of indices n. To illustrate the idea, take for example the case $p = 3$. In this case,

$$\omega_n = \begin{cases} 1 & n \equiv 0, 2 \bmod 3, \\ 2 & n \equiv 1 \bmod 3. \end{cases}$$

This can be seen by writing $n = 3t + j$, with $1 \leq j \leq 3$, and observing that

$$\omega_n = \prod_{k=1}^{t} (1+k^2)(1+(k+1)^2)(1+(k+2)^2) \times \prod_{k=3t+1}^{3t+j} (1+k^2).$$

The first factor is congruent to 1 modulo 3 and the result follows by considering the three cases for j. Therefore,

Corollary 5.6. *Assume $n \equiv 1 \bmod 3$. Then ω_n is not a square.*

Corollary 5.10 gives a full generalization of Corollary 5.6. In preparation, the sequence ω_n is analyzed modulo p.

Theorem 5.7. *Let $p \equiv 3 \bmod 4$ be a prime. Then the sequence*

$$\omega_{p,n} := \omega_n \bmod p,$$

is cyclic of period at most $\frac{p(p-1)}{2}$.

Proof. Since $p \equiv 3 \bmod 4$, the equation $1 + j^2 \equiv 0 \bmod p$ has no solution. On the other hand, for $1 \leq j \leq p$, the terms $1+j^2 \bmod p$ are symmetric with respect to p, that is, $1 + j^2 \equiv a \bmod p$ if and only if $1 + (p-j)^2 \equiv a \bmod p$. Therefore,

$$\begin{aligned}
\prod_{j=1}^{p(p-1)/2} (1+j^2) &\equiv \left(\prod_{j=1}^{p} 1+j^2 \right)^{(p-1)/2} \bmod p \equiv \left(\prod_{j=1}^{p-1} 1+j^2 \right)^{(p-1)/2} \bmod p \\
&\equiv \left(\prod_{j=1}^{(p-1)/2} (1+j^2)^2 \right)^{(p-1)/2} \bmod p \\
&\equiv \prod_{j=1}^{(p-1)/2} (1+j^2)^{p-1} \bmod p \\
&\equiv 1,
\end{aligned}$$

using Fermat's little theorem. Hence, the periodicity of ω_n modulo p is established. The period is (at most) $\binom{p}{2}$. □

Definition 5.8. *The Legendre symbol is defined by*

$$\left(\frac{a}{b}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue } \bmod b, \\ -1 & \text{if } a \text{ is not a quadratic residue } \bmod b, \\ 0 & \text{if } \gcd(a,b) > 1. \end{cases} \tag{5.14}$$

For a prime $p \equiv 3 \bmod 4$, define

$$\omega_{n,p}^{*} := \left(\frac{\omega_n}{p}\right) = \prod_{j=1}^{n} \left(\frac{1+j^2}{p}\right). \tag{5.15}$$

Observe that $1 + j^2 \not\equiv 0 \bmod p$, so $\omega_{n,p}^{*} \neq 0$.

Theorem 5.9. *Let p be a prime congruent to 3 modulo 4. The function $\omega_{p,n}^{*}$ is cyclic of period p. Moreover, in the list*

$$L_p := \left\{ \left(\frac{1+j^2}{p}\right) : \, 1 \leq j \leq p \right\}, \tag{5.16}$$

the number of -1 exceeds the number of $+1$ by 1.

Proof. The periodicity follows from that of the Legendre symbol. The sets

$$A := \left\{ (x,y) \in \mathbb{Z}_p \times \mathbb{Z}_p : \, 0 \leq x, \, y \leq p-1 \text{ and } 1 + x^2 = y^2 \right\}, \tag{5.17}$$

and

$$B := \{(u,v) \in \mathbb{Z}_p \times \mathbb{Z}_p : \, 0 \leq u, \, v \leq p-1 \text{ and } uv = 1\,\}, \tag{5.18}$$

have the same number of elements. Indeed, the map $(u,v) = (y-x, y+x)$ establishes a bijection from A to B and B has $p-1$ elements because

$$B = \{(u, u^{-1}) \text{ with } 1 \leq u \leq p-1\}.$$

In the set A, the value $y=0$ is excluded because $1+x^2 \equiv 0 \bmod p$ has no solutions. On the lattice $[0,p-1] \times [0,p-1] \subset \mathbb{Z}_p \times \mathbb{Z}_p$, mark the $p-1$ elements of the set A with a star. Every horizontal line (y fixed) contains either 0 or 2 stars. Thus, there must be $\frac{p-1}{2}$ horizontal lines with points of A, and each one produces two stars. But the lines corresponding to $+y$ and $-y$ produce the same pair, thus there are $\frac{p-1}{2}$ marked stars. These are precisely those for which the Legendre symbol of $1+x^2$ is $+1$. We conclude that the list L_p contains $\frac{p-1}{2}$ values $+1$ and $p - \frac{p-1}{2} = \frac{p+1}{2}$ values -1. □

Note. Fix a prime $p \equiv 3 \bmod 4$ and introduce the notation

$$\xi_j^p := \left(\frac{1+j^2}{p} \right). \tag{5.19}$$

Consider the sequence of partial products

$$\pi_k^p := \prod_{j=0}^{k} \xi_j^p, \quad k = 0, \, 1, \, 2, \, \cdots . \tag{5.20}$$

The periodicity of the Legendre symbol shows that the sequence $\{\pi_k^p : \, k \geq 0\}$ is also of period p. Moreover,

$$\pi_0^p = 1 \text{ and } \pi_{p-1}^p = 1, \tag{5.21}$$

given that there are an even number $(= \frac{p+1}{2}\,)$ of minus ones in the list L_p.

The next result can be employed to show that ω_n is not a square along certain arithmetic progressions.

Corollary 5.10. *Let $p \equiv 3 \bmod 4$ and assume $\pi_k^p = -1$. Then ω_n is not a square for $n \equiv k \bmod p$.*

Definition 5.11. *A* valid configuration *is a sequence of $+1$ and -1 of length p, with $\frac{p+1}{2}$ repetitions of -1 and $\frac{p-1}{2}$ of $+1$. It is also required that the sequence starts and end with $+1$.*

Theorem 5.12. *The minimum number of -1 in the sequence*

$$\Pi_p := \{\xi_k^p : \, 0 \leq k \leq p-1\} \tag{5.22}$$

is $\frac{p+1}{4}$. The maximum number is $\frac{3p-1}{4}$.

Proof. The minimum number is achieved when all the $\frac{p+1}{2}$ occurrences of -1 are at the right and this number is $\frac{p+1}{4}$. To prove this take a valid configuration and assume that it has a block of interior $+1$:

$$+1,\, \xi_2^p,\, \xi_3^p,\, \cdots,\xi_s^p,\, +1,\, +1,\, \xi_{s+3}^p,\, \xi_{s+4}^p,\, \cdots,\xi_{p-1}^p. \tag{5.23}$$

(where we have taken two internal $+1$ to illustrate the argument). Moving the (two) internal $+1$ to the left does not decrease the number of -1 in the product list Π_p. Indeed, if the partial product of the first s terms is $+1$, then the internal $+1$ simply repeat the $+1$. On the other hand, if the partial product is -1, then the internal $+1$ have the effect of repeating this -1, hence the total number of partial products equal to -1 increases.

The same argument shows that the maximum number of -1 in Π_p is $\frac{3p-1}{4}$. This occurs when all the -1 are aligned to the left of the $+1$. □

Corollary 5.13. *For each prime* $p \equiv 3 \bmod 4$*, there exist at least* $\frac{p+1}{4}$ *numbers* $k_i \in \{0, 1, 2, \cdots, p-1\}$ *such that* ω_n *is not a square for* $n \equiv k_i \bmod p$*. This yields a multi-infinite family of indices* n *for which* ω_n *is not square.*

Note. The total number of possible configurations of $+1$ and -1 is $\binom{p-1}{(p-1)/2}$. It would be of interest to explore how the $+1$ and -1 are distributed in Π_p as p varies. Figure 4 shows the proportion of -1 in Π_p, that is around $1/2$ for p large.

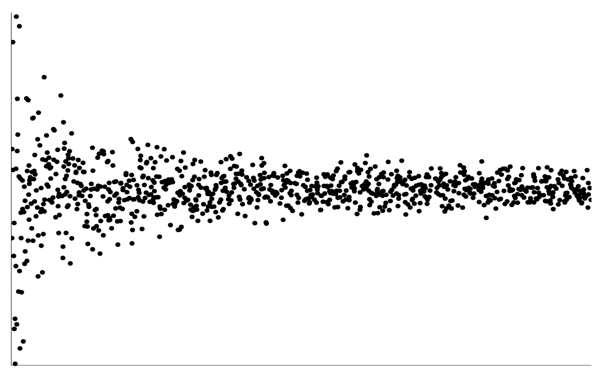

FIGURE 4. Proportion of minus ones for $6 \leq p \leq 3000$. The vertical range is $0.3 \leq y \leq 0.7$.

6. The p-adic valuation of ω_n

In this section we consider the p-adic valuation of ω_n. Our goal is to describe some relations between n and p that guarantees $\nu_p(\omega_n)$ is an odd integer.

Every odd prime divisor of ω_n is congruent to 1 modulo 4. We consider first the case $p = 2$ and then the odd primes. The case $p = 2$ admits a complete analytic solution. To evaluate $\nu_2(\omega_n)$, define

$$\mu_2(j) = \begin{cases} 0 & \text{if } j \equiv 0 \bmod 2, \\ 1 & \text{if } j \equiv 1 \bmod 2. \end{cases}$$

Proposition 6.1. *The* 2*-adic valuation of* ω_n *is given by*

$$\nu_2(\omega_n) = \lfloor \frac{n+1}{2} \rfloor. \tag{6.1}$$

Proof. From $\nu_2(1+j^2) = \mu_2(j)$, it follows that

$$\nu_2(\omega_n) = \sum_{j=1}^{n} \mu_2(j) = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} 1 = \lfloor \frac{n+1}{2} \rfloor.$$

□

Corollary 6.2. *Suppose* $n \equiv 1, 2 \bmod 4$*, then* ω_n *is not a square.*

Proof. For these values of n, the valuation $\nu_p(\omega_n)$ is odd. □

Combining the previous corollary with Corollary 5.6 yields a result modulo 12.

Corollary 6.3. *Suppose* $n \not\equiv 0, 3, 8, 11 \bmod 12$*, then* ω_n *is not a square.*

The next result employs the solutions to $x^2 + 1 \equiv 0 \bmod p$. This congruence has two solutions in the range $2 \leq x \leq p-1$. We denote by α_p the root that satisfies $2 \leq \alpha_p \leq \frac{p-1}{2}$. The other root is $\alpha_p^* = p - \alpha_p$. A simple argument shows the lower bound $\alpha_p \geq \sqrt{p-1}$. Moreover, this lower bound is achieved precisely when p is a prime of the form $1+n^2$.

Theorem 6.4. *Let* p *be a prime,* $p \equiv 1 \bmod 4$*. Assume* $n \in \mathbb{N}$ *lies in the range* $\alpha_p \leq n < p - \alpha_p$*. Then* ω_n *is not a square.*

Proof. In the product

$$\omega_n = \prod_{j=1}^{n} (1+j^2), \tag{6.2}$$

only the term corresponding to $j = \alpha_p$ is divisible by p. Moreover, since $1+n^2 < p^2$, we have $\omega_p(1+\alpha_p^2) = 1$. □

The previous theorem guarantees that ω_n is not a square for n in an interval of length $p - 2\alpha_p$. Therefore it is efficient for those primes p for which α_p is small. The distribution of α_p is a delicate question. We have computed the root α_p for primes of the form $p = 4m+1$ in the range $1 \leq m \leq 20000$. The ratio of α_p to its upper bound $2m+1$ attained its maximum value $38228/38367 \sim 0.996377$ at $m = 19183$ for the prime $p = 76733$. The minimum value $280/39201 \sim 0.00714267$ is achieved at $m = 19600$ for the prime $p = 78401$. This is the largest prime of the form $1 + n^2$ in the range considered.

The distribution of α_p is described in Figure 5. The graph shows the *normalized values*

$$\alpha_p^{nor} := \frac{\alpha_p - \sqrt{p-1}}{(p-1)/2 - \sqrt{p-1}}, \tag{6.3}$$

for $6 \leq p \leq 250000$. A result of W. Duke et al. [8], shows that these values are uniformly distributed on $[0, x] \times [0, 1]$ for large x.

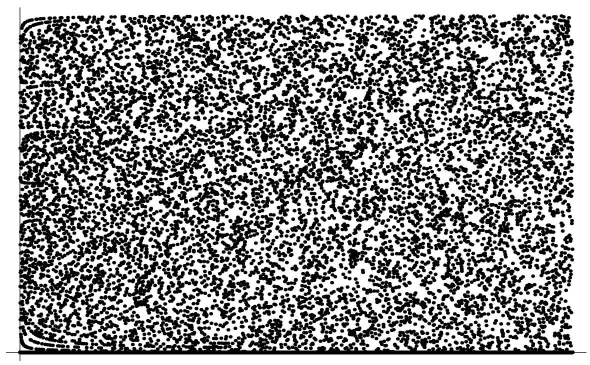

FIGURE 5. The values of α_p^{nor} for $6 \leq p \leq 250000$

Remark. Corollary 5.13 and Theorem 6.4 are a two-pronged approach in compiling evidence in favor of Conjecture 1.5. The former gives a successive list of infinite indices n, while the latter supplies endless interval ranges for n so that ω_n is not a square.

To each prime $p \equiv 1 \bmod 4$, associate the interval of $\mathbb{N}$ defined by

$$I_p := [\alpha_p, p - 1 - \alpha_p]. \tag{6.4}$$

Thus, if $n \in I_p$, then ω_n is not a square. The authors wish to thank N. Calkin for the sieve method used in the computations described in the next paragraph.

Conjecture 1.5 is now restated as

$$\bigcup_{p \equiv 1 \bmod 4} I_p = \mathbb{N} - \{3\}. \tag{6.5}$$

For notational simplicity, write $a_p = \alpha_p$ and $b_p = p - \alpha_p - 1$, so that $I_p = [a_p, b_p]$. In order to verify Conjecture 1.5 up to a certain bound n^*, it suffices to exhibit a sequence of primes $p_1, p_2, \cdots, p_k$ so that $4 \in I_{p_1}$, each interval I_{p_j} intersects the next one, and that $b_{p_k} > n^*$. Proceed as follows:

construct each p_{i+1} so that a_{p_i+1} is just below $b_{p_i} - 1$: the way to do this is to consider, for $j = 1, 2, \cdots$, the quantity $m^2 + 1$ where $m = p_i - a_{p_i} - j$: if there is a prime $q > 2m$, that divides $m^2 + 1$, then m is the smaller root of -1, namely a_q. Hence we may take $p_{i+1} = q$ and $a_{p_{i+1}} = m$.

In practice, we look for the largest prime q appearing as a factor of m^2+1 for the first 6 values of m less than $b_{p_i} - 1$.

Start with $p_1 = 17$ and check that $a_{p_1} = 4$ and $b_{p_1} = 12$. Therefore, the first interval is $I_{p_1} = [4, 12]$, contains 4 as required. Now consider numbers of the form $m := b_{p_1} - j = 12 - j$. The case $j = 2$ gives

$$(m-2)^2 + 1 = 101. \tag{6.6}$$

Therefore, $p_2 = 101$ and the second interval is $I_{p_2} = [10, 90]$. The process now continues with $m := 90 - j$ and, with $j = 6$, we find

$$(90-6)^2 + 1 = 7057. \tag{6.7}$$

We choose $p_3 = 7057$ and

$$I_{p_3} = [84, 6972]. \tag{6.8}$$

The list below provides the first six intervals. The chosen primes are $p_1 = 17$, $p_2 = 101$, $p_3 = 7057$, $p_4 = 48580901$, $p_5 = 1179713094952813$.

$$\begin{aligned} I_{p_1} &= [\,4, 12\,], \\ I_{p_2} &= [\,10, 90\,], \\ I_{p_3} &= [\,84, 6972\,], \\ I_{p_4} &= [\,6970, 48573930\,], \\ I_{p_5} &= [\,48573925, 1179713046378883]. \end{aligned}$$

Continuing this process, the next 8 more steps produce the following:

Computational fact. Assume the term ω_n is a square. Then either $n = 3$ or $n > 10^{3200}$.

Proposition 6.1 provides an exact formula for the 2-adic valuation of ω_n. The extension of this result for odd primes seems unlikely. We now establish an asymptotic result. Observe that

$$\omega_n = \prod_{j=1}^{n} (1 + j^2) = n!^2 \times \prod_{j=1}^{n} (1 + 1/j^2). \tag{6.9}$$

As $n \to \infty$ we have

$$\prod_{j=1}^{n} (1 + 1/j^2) \to \prod_{j=1}^{\infty} (1 + 1/j^2) = \frac{\sinh \pi}{\pi}.$$

We conclude that $\omega_n = O(n!^2)$. There is a famous result of Legendre [11, 12] for the p-adic valuation of $n!$. It states that

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1} \tag{6.10}$$

where $s_p(n)$ is the sum of the base$-p$ digits of n. In particular, $s_p(n) = O(\log_p n)$ as $n \to \infty$. Therefore

$$\nu_p(n!^2) \sim \frac{2n}{p-1}. \tag{6.11}$$

The same is true for $\nu_p(\omega_n)$.

Theorem 6.5. *Let p be an odd prime congruent to* $1 \bmod 4$. *Then*

$$\nu_p(\omega_n) \sim \frac{2n}{p-1}.$$

Proof. Consider first the contribution of α_p. Count the number of terms N_1 in the product for ω_n that are divisible by p. Recall that $1+j^2 \equiv 0 \bmod p$ if and only if $j \equiv \alpha$ or $\alpha^* = p - \alpha_p \bmod p$. Therefore, each interval of length p contains two such indices. The contribution of α_p is

$$N_1 = \lfloor \frac{n}{p} \rfloor + \begin{cases} 1 & \text{if } \alpha_p + \lfloor \frac{n}{p} \rfloor p \leq n \\ 0 & \text{if } \alpha_p + \lfloor \frac{n}{p} \rfloor p > n. \end{cases} \tag{6.12}$$

Therefore $N_1 \geq \lfloor \frac{n}{p} \rfloor$. Similarly, by considering the elements α_{p^i} described (1.25), one sees that the number of terms in $[1, n]$ divisible by p^i is at least $\lfloor \frac{n}{p^i} \rfloor$. Therefore, the contribution of α_p to $\nu_p(\omega_n)$, denoted by $\nu_p(\omega_n, \alpha_p)$, is at least

$$\nu_p(\omega_n, \alpha_p) \geq \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor = \sum_{k=1}^{z_{p,n}} \lfloor \frac{n}{p^k} \rfloor,$$

where $z_{p,n} = \lfloor \log_p n \rfloor$. Now

$$\begin{aligned} \nu_p(\omega_n, \alpha_p) &\geq \sum_{k=1}^{z_{p,n}} \lfloor \frac{n}{p} \rfloor \geq \sum_{k=1}^{z_{n,p}} \left(\frac{n}{p^k} - 1 \right) \\ &= n \left(\frac{1 - p^{-1-z_{p,n}}}{1 - 1/p} - 1 \right) - z_{p,n} \geq n \left(\frac{1 - 1/n}{1 - 1/p} - 1 \right) - z_{p,n} \\ &= \frac{n-p}{p-1} - z_{p,n}. \end{aligned}$$

Thus

$$\frac{p-1}{n} \nu_p(\omega_n, \alpha_p) \geq 1 - \frac{p}{n} - \frac{p-1}{n} z_{p,n}, \tag{6.13}$$

and it follows that

$$\liminf_{n \to \infty} \frac{p-1}{n} \nu_p(\omega_n, \alpha_p) \geq 1. \tag{6.14}$$

The same holds for the contribution from α_p^*. We conclude that

$$\liminf_{n\to\infty} \frac{p-1}{2n}\nu_p(\omega_n) \geq 1. \tag{6.15}$$

To obtain an upper bound, observe again that $\nu_p(1+j^2)=0$ unless $j \equiv \alpha_p$ or α_p^* modulo p. Define

$$\tau_n := \prod_{k=1}^{n} (1+(pk+\alpha_p)^2) \times (1+(pk+\alpha_p^*)^2). \tag{6.16}$$

The bounds on α_p show that $1+\alpha_p^2 = pb_1$ with $b_1 \not\equiv 0 \bmod p$. Write

$$1+(pk+\alpha_p)^2 = pf(k), \tag{6.17}$$

with

$$f(k) = b_1 + 2\alpha_p k + pk^2, \tag{6.18}$$

and conclude that

$$\nu_p(\tau_n) = 2(n+1) + \sum_{k=0}^{n} \nu_p(f(k)) + \sum_{k=0}^{n} \nu_p(f^*(k)), \tag{6.19}$$

where $f^*(k)$ is formed from α_p^* as f was from α_p.

Define

$$r(n) := \text{Max}\left\{ j:\ p^j \text{ divides } f(k) \text{ for some } k \in \{1,\, 2,\, \cdots, n\} \right\}, \tag{6.20}$$

and let N_i be the number of terms in the sum (6.19) such that $f(k)$ is divisible by p^i. Then

$$\begin{aligned}
\sum_{k=0}^{n} \nu_p(f(k)) &= N_1 + N_2 + \cdots + N_{r(n)} \\
&\leq r(n) + \sum_{i=1}^{\infty} \lfloor \frac{n}{p^i} \rfloor \\
&\leq r(n) + \frac{n}{p-1}.
\end{aligned}$$

Taking into account the contribution of α_p^* we obtain

$$\nu_p(\tau(n)) \leq 2(n+1) + 2r(n) + \frac{2n}{p-1}. \tag{6.21}$$

To obtain the estimate for $\nu_p(\omega_n)$, observe that

$$\nu_p(\omega_{pn}) = \nu_p(\tau_{n-1}). \tag{6.22}$$

Now use $n \leq Np$ with $N := \lfloor \frac{n}{p} \rfloor + 1$ and since $|f(k)| \leq Ck^3$ shows that $p^{r(n)} \leq Cn^3$, then

$$\begin{aligned} \nu_p(\omega_n) &\leq \nu_p(\omega_{Np}) = \nu_p(\tau_{N-1}) \\ &\leq 2\left(\lfloor \frac{n}{p} \rfloor + 1\right) + 2r\left(\lfloor \frac{n}{p} \rfloor\right) + \frac{2\lfloor \frac{n}{p} \rfloor}{p-1} \\ &\leq \frac{2n}{p-1} + 2 + 2r\left(\lfloor \frac{n}{p} \rfloor\right). \end{aligned}$$

We conclude that

$$\limsup_{n \to \infty} \frac{p-1}{2n} \nu_p(\omega_n) \leq 1. \tag{6.23}$$

□

Remark 1. The error term

$$\text{error}_p(n) := \nu_p(\omega_n) - \frac{2n}{p-1},$$

in Theorem 6.5 is shown in Figure 6 for $p = 29$ and $1 \leq n \leq 10^6$. Figure 7 shows the difference between $\nu_p(\omega_n)$ and $\nu_p(n!^2)$ for the same values of n. These two functions have the same asymptotic behavior and $\nu_p(n!^2)$ acts as a stabilizing factor by absorbing the fluctuations. The patterns appearing in this error terms have certain structure that deserves to be elucidated.

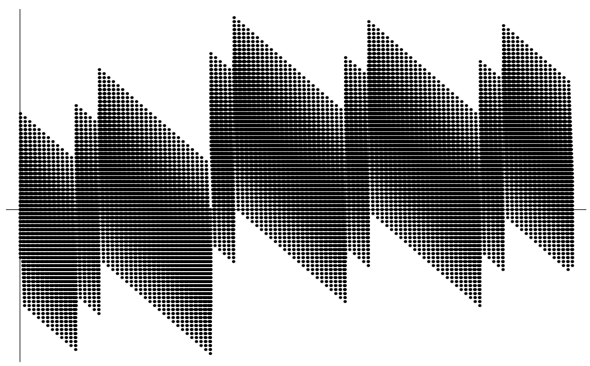

FIGURE 6. The error term in the 29-adic valuation of ω_n.

Remark 2. The polynomial f, appearing in (6.18), satisfies $f(k) \equiv b_1 + 2\alpha_p k \bmod p$. Therefore there is a unique solution to the congruence $f(k) \equiv 0 \bmod p$. Moreover, $f'(k) \equiv 2\alpha_p \not\equiv 0 \bmod p$. Hensel's lemma [10] guarantees the existence of $\bar{\beta} \in \mathbb{Z}_p$ such that $f(\bar{\beta}) = 0$ in $\mathbb{Q}_p$. The number $\bar{\beta}$ is written as

$$\bar{\beta} = \beta_0 + \beta_1 p + \beta_2 p^2 + \cdots. \tag{6.24}$$

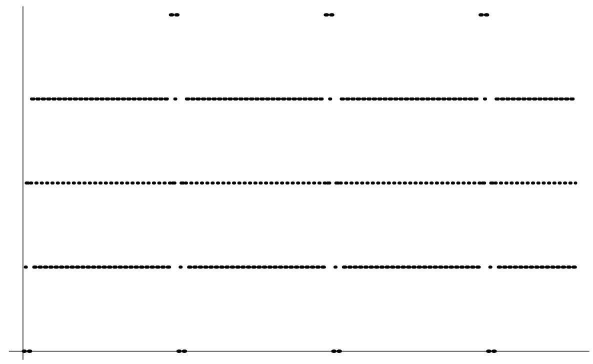

FIGURE 7. The error term $\nu_{29}(\omega_n) - \nu_{29}(n!^2)$.

Moreover,

$$f(k) \equiv 0 \bmod p^i \text{ if and only if } k \equiv \sum_{m=0}^{i-1} \beta_m p^m \bmod p^i. \tag{6.25}$$

Introduce the notation

$$\gamma(i,p) = \beta_0 + \beta_1 p + \cdots + \beta_{i-1}p^{i-1}, \tag{6.26}$$

and conclude that

$$\sum_{k=0}^{n} \nu_p(f(k)) = \sum_{i=1}^{r(n)} \sum_{k \equiv \gamma(i,p) \bmod p}^{n} 1.$$

The fact is that

$$N_i = \sum_{k \equiv \gamma(i,p) \bmod p}^{n} 1. \tag{6.27}$$

This point of view yields a more general result. Details will be presented elsewhere.

Theorem 6.6. *Let P be a polynomial with integer coefficients and without integer roots. Define*

$$z_p := |\{b \in \{0, 1, 2, \cdots, p-1\} : P(b) \equiv 0 \bmod p\}|. \tag{6.28}$$

Assume that all the z_p roots satisfy the hypothesis of Hensel's lemma. Then the recurrence $t_n := P(n)t_{n-1}$, with $t_0 = 1$ satisfies

$$\nu_p(t_n) \sim \frac{z_p n}{p-1} \quad \text{as } n \to \infty. \tag{6.29}$$

The next result establishes a connection between ω_n and primes of the form $1 + m^2$. The authors wish to thank C. Pommerance for providing this result.

Theorem 6.7. *Suppose that for $n \in \mathbb{N}$ there exists an integer x_0 such that $\lfloor\sqrt{n}\rfloor + 2 \leq x_0 \leq n$ and $p = 1 + x_0^2$ is a prime. Then ω_n is not a square.*

Proof. We show that the prime p appears with exponent 1 in the product ω_n. The congruence $1 + x^2 \equiv 0 \bmod p$ has two solutions α_p, $p - \alpha_p$. The bounds on x_0 imply that $x_0 = \alpha_p$. It follows that $\lfloor\sqrt{n}\rfloor + 2 \leq \alpha_p \leq n$. Then the other root $p - \alpha_p$ is bigger than n because $\alpha_p^2 - \alpha_p + 1 - n > 0$. To check this inequality observe that the largest root of $x^2 - x + 1 + n = 0$ is $(1 + \sqrt{4n-3})/2$ and

$$\alpha_p > \sqrt{n} + 1 > \frac{1}{2}\left(1 + \sqrt{4n-3}\right).$$

To conclude the proof, observe that any other factor in ω_n that produces a multiple of p must be of the form $\alpha_p + mp$. But

$$\alpha_p + p = p - \alpha_p + 2\alpha_p = p + \alpha_p > n,$$

so they are outside the range $4 \leq j \leq n$. □

The previous theorem can be improved by relaxing the condition that $1 + x^2$ is a prime.

Proposition 6.8. *Suppose that for $n \in \mathbb{N}$ there exists a prime p, a real number $c_n \in (0, 1]$ and positive integers x, y, with y odd, such that*

$$(1 + c_n^{-1})x \leq p, \; nc_n < x \leq n, \; \textit{and } \nu_p(1 + x^2) = y. \tag{6.30}$$

Then ω_n is not a square.

Proof. The condition $x \leq n$ shows that p^y divides ω_n. The hypothesis imply that x is one of the solutions to $1 + x^2 \equiv 0 \bmod p$. The other solution is $p - x \geq c_n^{-1}x > n$, so this term does not contribute to the product ω_n. It follows that $\nu_p(\omega_n) = y$. The fact that y is odd, shows that ω_n is not a square. □

7. Miscellaneous

In this section we present several problems inspired by the results presented in this paper.

7.1. Connections with triangular numbers.

Splitting the product

$$\omega_n = \prod_{j=1}^{n} (1 + j^2) \tag{7.1}$$

according to the parity of the index j produces

$$\prod_{j=1}^{n}(1 + j^2) = 2^{\lfloor (n-1)/2 \rfloor - 1} \prod_{k=1}^{\lfloor n/2 \rfloor} (1 + 4k^2) \times \prod_{k=1}^{\lfloor (n-1)/2 \rfloor} (1 + 4\Delta(k)), \tag{7.2}$$

where

$$\Delta(k) = \frac{k(k+1)}{2} \tag{7.3}$$

is the k-th triangular number.

Conjecture 7.1. *The even and odd parts of ω_n are defined by*

$$t_n := \prod_{k=1}^{n} (1+2k(k-1)), \text{ and } s_n := \prod_{k=1}^{n} (1+4k^2). \tag{7.4}$$

These products involve the triangular and square numbers respectively. Neither of them is a perfect square.

We now present a problem describing a connection between triangular numbers and primes of the form $1+x^2$.

Conjecture 7.2. *Assume $n \in \mathbb{N}$ and $n \neq 27, 35$. Then there exists an index x, such that $\Delta_n \leq x < \Delta_{n+1}$ and $1+x^2$ is prime.*

Note. The authors wish to thank Dante Manna, who verified this conjecture up to $n = 10^6$.

The next statement is the result of our study of the set of *square-triangular* numbers:

$$U := \{1 + 4\Delta_k : \Delta_k \text{ is a square}\}. \tag{7.5}$$

Proposition 7.3. *Let $x = \Delta_k$ be a square triangular number, i.e., $s := 1+4x \in U$. Then*
a) $(s-1)(2s-1)$ is a perfect square.
b) s is not a prime, unless $s = 5$.

Proof. Part a) is elementary: $(s-1)(2s-1) = 4j^2(2k+1)^2$, where $x = \Delta_k = j^2$. To prove b), assume s is prime and observe from a) that $s(2s-3) = (j-1)(j+1)$. If s divides $j-1$, we have $s(2s-3) = sb(sb+2)$, for some $b \in \mathbb{N}$. This is valid only if $s = 5$. On the other hand, if s divides $j+1$, we have $2s - 3 = c(sc-2)$. An elementary argument shows that this is impossible. □

Note. Part (b) of Proposition 7.3 informs us that identical entries in the two products from (7.2) can not produce the same primes.

7.2. **Connections with Stirling numbers.** The *Stirling numbers of the first kind* are given by

$$\prod_{k=1}^{n} (1+kx) = \sum_{k=1}^{n+1} (-x)^{n+1-k} s(n+1,k). \tag{7.1}$$

It follows that

$$S_+(n) + iS_-(n) = \sum_{k=1}^{n+1} i^{k-1} s(n+1,k). \tag{7.2}$$

Introduce the notation

$$C_j(n) := \sum_{k\geq 0} |s(n+1, 4k+j)| \tag{7.3}$$

for $0 \leq j \leq 3$. The number $C_j(n)$ counts the total number of permutations of $\{1, 2, \ldots, n+1\}$, which contain exactly $4k+j$ cycles, $k \geq 0$.

The statements below provide a combinatorial interpretation of Conjecture 1.2 as well as consequences of our established results.

Proposition 7.4. *The symmetric functions $S_\pm(n)$ are given by*

$$\begin{aligned}
(-1)^n S_+(2n) &= C_1(2n) - C_3(2n) \\
(-1)^n S_-(2n) &= C_0(2n) - C_2(2n) \\
(-1)^{n+1} S_+(2n+1) &= C_0(2n+1) - C_2(2n+1) \\
(-1)^n S_-(2n+1) &= C_1(2n+1) - C_3(2n+1).
\end{aligned}$$

Proposition 7.5. *The problem of whether x_n or $1/x_n$ is an integer is equivalent to finding $n \in \mathbb{N}$ such that either $C_0 - C_2$ divides $C_1 - C_3$, or vice-versa.*

For example, it is clear that $C_0 + C_2 = C_1 + C_3 = n!/2$. Theorem 2.1 and its Corollary 2.7 show the following result.

Corollary 7.6. *$C_0 \neq C_2$ and $C_1 \neq C_3$ for $n \geq 5$. Also $C_0 - C_2 \neq n(C_1 - C_3)$ and $C_1 - C_3 \neq n(C_2 - C_0)$.*

7.3. **The bound $|x_n| \leq n$.** In this section we prove that the even and odd subsequence of x_n, namely $\{x_{2n}\}$ and $\{x_{2n+1}\}$ satisfy the bounds $|x_{2n}| \leq 2n$ and similarly $|x_{2n+1}| \leq 2n+1$ for almost all $n \in \mathbb{N}$. The exceptions are described below. We give the details for x_{2n}.

The parity dependent identities (3.8) show that

$$x_{2n} = \tan\left(- \sum_{k=1}^{2n} \tan^{-1} \frac{1}{k} \right). \tag{7.1}$$

The sequence x_{2n} begins in a decreasing fashion:

$$\{4, \frac{105}{73} \sim 1.4383, \frac{36}{43} \sim 0.837209, \frac{2387}{4511} \sim 0.529151, \frac{104472}{322921} \sim 0.323522 \cdots \}.$$

This continues until the angle

$$-\sum_{k=1}^{2n} \tan^{-1} \frac{1}{k} > \frac{\pi}{2}, \tag{7.2}$$

so that the sequence jumps to the next branch of the tangent function. For each $j \in \mathbb{N}$ define the *transition points*

$$\kappa_j^+ := \text{Inf}\left\{ N \in \mathbb{N} : \; -\sum_{k=1}^{2N} \tan^{-1}\frac{1}{k} > (2j-1)\frac{\pi}{2} \right\}. \tag{7.3}$$

The divergence of the series $\sum \tan^{-1} 1/k$ guarantees the existence of the sequence

$$\kappa := \{\kappa_1^+, \, \kappa_2^+, \, \kappa_3^+, \cdots \}. \tag{7.4}$$

Conjecture 7.7. *There exits a constant κ_∞ such that the sequence κ_j^+ grows quasi-geometrically as κ_∞^{j-1}. Numerical calculations show that $\kappa_\infty \sim 23.1$.*

Define the interval

$$I_j := \{ m \in \mathbb{N} : \; \kappa_j^+ \leq m < \kappa_{j+1}^+ \}. \tag{7.5}$$

The construction of the transition points immediately gives the next result:

Lemma 7.8. *Fix $j \in \mathbb{N}$. Then the sequence $\{x_{2n} : \, n \in I_j\}$ is decreasing.*

Corollary 7.9. *Let $n, \, m \in I_j$ and $n \neq m$. Then $x_n \neq x_m$.*

We now establish the promised bound.

Theorem 7.10. *Fix $j \in \mathbb{N}$. Then, for every n in the range $\kappa_j^+ + 1 \leq n \leq \kappa_{j+1}^+ - 2$, we have $|x_{2n}| \leq n+1$.*

Proof. The sequence $\{x_{2n} : \, n \in \mathbb{N}\}$ satisfies the recurrence

$$x_{2n+2} = \frac{a \cdot x_{2n} - b}{b \cdot x_{2n} + a}, \tag{7.6}$$

where $a = 2(2n+1)(n+1) - 1$ and $b = 4n+3$. This follows by iteration of (1.5). The proof of the bound is divided in cases according to the sign of x_{2n}.

Case 1. If $x_{2n+2} > 0$, then $x_{2n} > x_{2n+2} > 0$ by Lemma 7.8. The result now follows from

$$x_{2n+2} = \frac{a - b/x_{2n}}{b + a/x_{2n}} < \frac{a}{b} \leq n+1, \tag{7.7}$$

and the base case $x_{2\kappa_j^+} > 0$.

Case 2. If $x_{2n-2} < 0$, then $x_{2n} < 0$. We now take $x_{2\kappa_{j+1}^+ - 2} < 0$ as the base case and work backwards. Define $y_{2n} := -x_{2n}$. Then (7.7) gives

$$y_{2n-2} = |x_{2n-2}| = \frac{c \cdot y_{2n} - d}{d \cdot y_{2n} + c}, \tag{7.8}$$

with $c := 2n(2n-1) - 1$ and $d := 4n-1$. The same argument given in Case 1 now yields $|x_{2n-2}| \leq n$.

In both cases we get the bound $|x_{2n}| \leq n+1$. ∎

Corollary 7.11. *Assume* $n \notin \kappa$. *Then* $|x_{2n}| \leq n+1$. *A similar conclusion can be drawn for the odd terms.*

7.4. **The p-adic valuation of x_n.** It might be possible to extend the results on $\nu_2(x_n)$ to odd prime valuations. Some information about the case $p=3$ is given next. Extensive symbolic calculations suggest that

$$\nu_3(x_n) = 0, \tag{7.1}$$

precisely when $n \geq 5$ and $n \equiv 1 \bmod 3$. Similar conjectures can be made for the set

$$\tau_{3,1} := \{n \in \mathbb{N} : \, \nu_3(x_n) = 1\} = \{6,\, 11,\, 15,\, 20,\, 24, \cdots\}. \tag{7.2}$$

We have observed that the difference set

$$\tau_{3,1}^{+} := \{\tau_{3,1}(n+1) - \tau_{3,1}(n) : \, n \geq 5\}, \tag{7.3}$$

is the periodic sequence

$$\tau_{3,1}^{+} = \{\mathbf{5}, \mathbf{4}\} = \{5,\, 4,\, 5,\, 4, \cdots\}. \tag{7.4}$$

Similarly

$$\begin{aligned} \tau_{3,2}^{+} &= \{\mathbf{3},\, \mathbf{1},\, \mathbf{3},\, \mathbf{2},\, \mathbf{3},\, \mathbf{1},\, \mathbf{3},\, \mathbf{11}\}, \\ \tau_{3,3}^{+} &= \{\mathbf{3},\, \mathbf{1},\, \mathbf{3},\, \mathbf{20},\, \mathbf{3},\, \mathbf{1},\, \mathbf{3},\, \mathbf{47}\}, \\ \tau_{3,4}^{+} &= \{\mathbf{3},\, \mathbf{1},\, \mathbf{3},\, \mathbf{74},\, \mathbf{3},\, \mathbf{1},\, \mathbf{3},\, \mathbf{155}\}, \end{aligned}$$

where we have only indicated the period.

There is a marked difference in the behavior according to whether $p \equiv 1 \bmod 4$ or $3 \bmod 4$. Figure 8 shows $\nu_3(x_n)$ and Figure 9 shows $\nu_5(n)$.

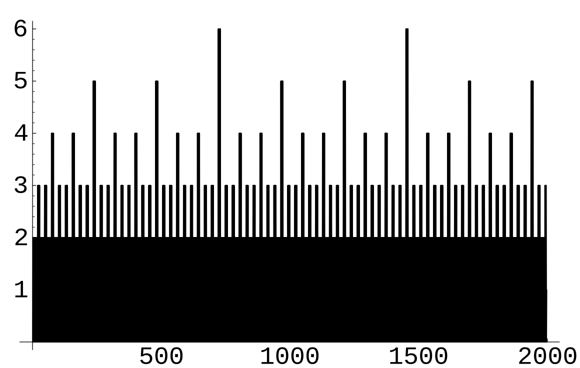

FIGURE 8. The 3-adic valuation of x_n

An argument similar to the proof of Theorem 2.2 yields the next result. The statement was found by examining the data given in the list $\tau_{3,s}^{+}$ described above.

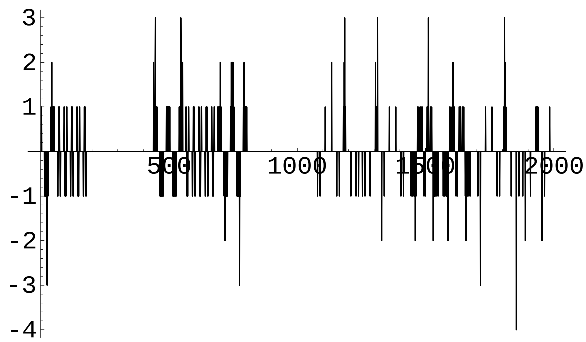

FIGURE 9. The 5-adic valuation of x_n

Theorem 7.12. *The 3-adic valuation of x_n is given by*

$$\nu_3(x_n) = \nu_3(n(n+1)) + \delta_{9\mathbb{Z}+5,n} \cdot \nu_3\left(\lfloor \frac{n+4}{3} \rfloor\right) + \delta_{9\mathbb{Z}+3,n} \cdot \nu_3\left(3\lfloor \frac{n+3}{9} \rfloor\right).$$

Here $\delta_{A,n}$ is the Knonecker delta: 1 if $n \in A$ and 0 otherwise.

Once again, the next result can be established as in the case $p = 2$.

Proposition 7.13. *The even partial sums satisfy $\nu_3(S_+(n)) = 0$ and the odd ones $\nu_3(S_-(n)) = \nu_3(x_n)$.*

7.5. **Geometric properties of the sequence x_n.** The representation

$$x_n = \frac{S_-(n)}{S_+(n)}, \tag{7.1}$$

established in Theorem 3.6 has a geometric interpretation. We consider the map

$$\eta(n) := (S_+(n),\, S_-(n)). \tag{7.2}$$

The point $\eta(n)$ has modulus ω_n and the sequence

$$\frac{\omega_n}{n!^2} = \prod_{j=1}^{n} \left(1 + \frac{1}{j^2}\right) \tag{7.3}$$

converges from below to its limit $\frac{\sinh \pi}{\pi}$.

Define

$$a_+(n) := \frac{S_+(n)}{n!}, \quad a_-(n) := \frac{S_-(n)}{n!}. \tag{7.4}$$

Naturally, $x_n = a_-(n)/a_+(n)$. We consider the generating functions

$$A_+(x) := \sum_{n=1}^{\infty} a_+(n)x^n, \quad A_-(x) := \sum_{n=1}^{\infty} a_-(n)x^n. \tag{7.5}$$

Lemma 7.14. *The sequences $a_\pm(n)$ satisfy the discrete dynamical system*

$$\begin{aligned}(n+1)a_-(n+1) - a_-(n) &= (n+1)a_+(n), \\ (n+1)a_+(n+1) - a_+(n) &= -(n+1)a_-(n),\end{aligned} \tag{7.6}$$

with initial conditions $a_+(1) = 1$, $a_+(2) = -1$, $a_-(1) = 1$, $a_-(2) = 3$. Therefore, the generating functions are given by

$$A_+(x) = \frac{e^{\tan^{-1}x}}{1+x^2}\left(x\,\cos(\log(\sqrt{1+x^2})) + \sin(\log\sqrt{1+x^2})\right),$$
$$A_-(x) = \frac{e^{\tan^{-1}x}}{1+x^2}\left(x\,\cos(\log(\sqrt{1+x^2})) - x\sin(\log\sqrt{1+x^2})\right),$$

Thus, the pair $(A_+(x), A_-(x))$ forms a spiral in the complex plane, running inward towards the origin.

Proof. The recurrences (7.6) show that $A_+(x)$ and $A_-(x)$ both solve the second order differential equation

$$(1+x)(1+x^2)D^2y + (3x^2+2x-3)Dy + 2(x+2)y = 0. \tag{7.7}$$

Standard techniques produce the analytic solutions given above. $\square$

7.6. **A connection with Euler's constant.** The claim in this section corresponds to an analogue of Proposition 5.1. More precisely, the proof of the above-mentioned proposition exploits the existence of a prime between an integer and its double (this is Bertrand's postulate). In the same spirit, our claim highlights a prime p between n and $1+n^2$, for which $\nu_p(\omega_n) = 1$, that is, p divides ω_n but p^2 does not. The conclusions described in this section are by-in-large empirical and the arguments are heuristic.

Section 7.5 shows that the expressions

$$\omega_n = (1+1^2)(1+2^2)(1+3^2)\cdots(1+n^2), \tag{7.1}$$

and $n!^2$ are of comparable size. Morever, Theorem 6.5 establishes that the p-adic valuations of these two terms have the same asymptotic behavior. Naturally, *every* prime $p < n$ divides $n!$, but only primes $p \equiv 1 \bmod 4$ divide ω_n. Therefore, ω_n is missing (essentially) half the primes of $n!^2$.

Denote by $\mathbb{P} := \{p_1 < p_2 < p_3 \cdots\}$ be the complete set of primes, and $\mathbb{P}^{(1)} := \{q_1 < q_2 < q_3 \cdots\}$ be those primes $q_i \equiv 1 \bmod 4$. The classical prime number theorem shows that $p_n \sim n\,\log n$, and P. Dusart [9] proved that

$$n\log n + n\log\log n - n < p_n < n\log n + n\log\log n,\; n \geq 2. \tag{7.2}$$

The proof is based on the knowledge of the first 1.5 billion zeros of the Riemann zeta function $\zeta(s)$, that lie on the critical line $\text{Re}\, s = \frac{1}{2}$. Using the equi-distribution of primes $\mathbb{P}^{(1)}$ in $\mathbb{P}$ to conclude that

$$2n\log 2n + 2n\log\log 2n - 2n < q_n < 2n\log 2n + 2n\log\log 2n, \tag{7.3}$$

at least for large values of n.

The objective is now to produce a sequence of indices $y(n)$ so that q_n divides $\omega_{y(n)}$, but q_n^2 does not. In order to accomplish this, observe first that, if q is a prime such that $m < q < 1+m^2$, then $\nu_q(\omega_m) \leq 2$. In fact, $\nu_q(\omega_m) = 2$ if and only if both $\alpha_q, \, \alpha_q^* \leq m$.

The inequalities (7.3) suggest that we choose m around $2n \log 2n$. In order to fine-tune the constant in $m = C_3 n \log n$, we make use of the inequalities

$$\sqrt{C_1} m! < \sqrt{\omega_m} < \sqrt{C_2} m!, \tag{7.4}$$

with $C_1 \geq \frac{5}{2}$ and $C_2 \leq \frac{\sinh \pi}{\pi} \sim 3.676$. The identity

$$\frac{\sinh \pi}{\pi} = \lim_{k \to \infty} \prod_{j=1}^{k} \left(1 + \frac{1}{j^2} \right) \tag{7.5}$$

and the observation

$$\prod_{j=1}^{k} \left(1 + \frac{1}{j^2} \right) \sim 1 + H_k^{(2)}, \tag{7.6}$$

where $H_k^{(2)}$ is the second harmonic number, lead to

$$\sqrt{H_k^{(2)}} \sim H_k^{(1)} \sim \log k + \gamma, \tag{7.7}$$

where γ is Euler's constant defined by

$$\gamma := \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \log n. \tag{7.8}$$

The logarithmic part has been absorbed, so we consider $m = y(n) \sim \gamma n \log n$. Numerical experiments suggested the extra factor $\sqrt{5}$ in the next statement.

Theorem 7.15. *Define $y(n) := \lfloor \sqrt{5} \, \gamma \, n \log n \rfloor$. Then, for almost all $n \in \mathbb{N}$, we have*

$$\nu_{q_n}(\omega_{y(n)}) = 1. \tag{7.9}$$

Finally, consider the intervals $J_k := [y(k), y(k+1) \,)$, with $y(k)$ as above. This yields a partition of $\mathbb{N}$ in the form

$$\mathbb{N} = \bigcup_{k \geq 2} J_k. \tag{7.10}$$

Given $n \in \mathbb{N}$, there is a unique k such that $n \in J_k$. Define the map

$$\Phi(n) = \begin{cases} \nu_{q_k}(\omega_n) & \text{if } \nu_{q_k}(\omega_{y_k}) = 1, \\ \nu_{q_{k-1}}(\omega_n) & \text{if } \nu_{q_k}(\omega_{y_k}) = 0, \\ \nu_{q_{k+1}}(\omega_n) & \text{if } \nu_{q_k}(\omega_{y_k}) = 2. \end{cases} \tag{7.11}$$

The previous theorem guarantees that almost all cases correspond to the first choice in (7.11). The other two cases rectify the exceptions. The last two assignments are implicitly guided by the *prime gaps* to the effect that

$$p_{N+1} - p_N = O\left(\sqrt{p_N}\,\log p_N\right). \tag{7.12}$$

H. Cramer [7] proved (7.12) assuming the validity of the Riemann hypothesis.

Conjecture 7.16. *For $n \geq 4$, we have $\Phi(n) = 1$. Hence, ω_n is not a square.*

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