A NOTE ON A LEMMA IN THE ARTICLE: A DIVISIBILITY APPROACH TO THE OPEN BOUNDARY CASES OF CUSICK-LI-STĂNICĂ'S CONJECTURE

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ABSTRACT. In this note we present a proof of Lemma 3.3 in the article: "*A divisibility approach to the open boundary cases of Cusick-Li-StĂnicĂ's conjecture*".

1. Proof of Lemma 3.3

This note is devoted to the proof Lemma 3.3. We re-state the lemma in order to aid the reader.

Lemma 3.3. *Let a, i, m be natural numbers with $i \geq 3$ and m odd. Write $m = b_s \cdot 2^s + b_{s-1} \cdot 2^{s-1} + \cdots + b_1 \cdot 2 + 1$ with $s \geq 1$. Let $b_{s-l_1}, b_{s-l_2}, \ldots, b_{s-l_r}$ be all the b_t in the expansion of m such that $b_t = 0$. Define*

$$b_{a,i} = \delta_1 \cdot 2^{a+i-1+(s-l_1)} + \cdots + \delta_r \cdot 2^{a+i-1+(s-l_r)}$$

Then,

$$\frac{(2^{a+i-1} \cdot m - 2^a + b_{a,i} + 2^{a+i-1}) \cdots (2^{a+i-1} \cdot m - 2^a + b_{a,i} + 1)}{(2^{a+i-1} \cdot m - b_{a,i}) \cdots (2^{a+i-1} \cdot m - b_{a,i} - 2^{a+i-1} + 1)} \equiv 3 \bmod 4. \tag{1}$$

Proof. The proof we present is elementary, but rather long. We decided to present most of the details, including a double induction, because this technique can be used to prove other results.

Note that the left hand side of (1) is an odd number, which implies that the numbers of even terms in the numerator and the denominator are equal. This will allow us to perform reductions modulo 8 in the numerator and denominator independently. Also note that the number of terms in the numerator and denominator is 2^{a+i-1}. We prove (1) by double induction.

Suppose first that $i = 3$. We want to prove that

$$\frac{(2^{a+2} \cdot m - 2^a + b_{a,3} + 2^{a+2}) \cdots (2^{a+2} \cdot m - 2^a + b_{a,3} + 1)}{(2^{a+2} \cdot m - b_{a,3}) \cdots (2^{a+2} \cdot m - b_{a,3} - 2^{a+2} + 1)} \equiv 3 \bmod 4$$

for $a \geq 1$. For the base case, i.e. $a = 1$, we have

$$\begin{aligned}
&\frac{(2^3 \cdot m - 2 + b_{1,3} + 2^3) \cdots (2^3 \cdot m - 2 + b + 1)}{(2^3 \cdot m - b_{1,3}) \cdots (2^3 \cdot m - b_{1,3} - 2^3 + 1)} \\
&= \frac{(2^3 \cdot m + b_{1,3} + 6) \cdots (2^3 \cdot m + b_{1,3} - 1)}{(2^3 \cdot m - b_{1,3}) \cdots (2^3 \cdot m - b_{1,3} - 7)} \\
&= \frac{(2^3 \cdot m + b_{1,3} + 6) \cdots (2^3 \cdot m + b_{1,3} + 1) \cdot (2^3 \cdot m + b_{1,3} - 1)}{(2^3 \cdot m - b_{1,3} - 1) \cdots (2^3 \cdot m - b_{1,3} - 7)} \cdot \frac{(2^3 \cdot m + b_{1,3})}{(2^3 \cdot m - b)} \\
&\equiv \frac{(4 \cdot m + 3) \cdot (2 \cdot m + 1) \cdot (4 \cdot m + 1)}{(4 \cdot m - 1) \cdot (2 \cdot m - 1) \cdot (4 \cdot m - 3)} \cdot \frac{(m + \delta_1 \cdot 2^{(s-l_1)} + \cdots + \delta_r \cdot 2^{(s-l_r)})}{(m - \delta_1 \cdot 2^{(s-l_1)} - \cdots - \delta_r \cdot 2^{(s-l_r)})} \bmod 8 \\
&\equiv 3 \bmod 4.
\end{aligned}$$

Thus the claim holds for $a = 1$. Suppose that

$$\frac{(2^{a+2} \cdot m - 2^a + b_{a,3} + 2^{a+2}) \cdots (2^{a+2} \cdot m - 2^a + b_{a,3} + 1)}{(2^{a+2} \cdot m - b_{a,3}) \cdots (2^{a+2} \cdot m - b_{a,3} - 2^{a+2} + 1)} \equiv 3 \bmod 4 \tag{2}$$

for some $a \geq 1$. We will show that

$$\frac{(2^{a+3} \cdot m - 2^{a+1} + b_{a+1,3} + 2^{a+3}) \cdots (2^{a+3} \cdot m - 2^{a+1} + b_{a+1,3} + 1)}{(2^{a+3} \cdot m - b_{a+1,3}) \cdots (2^{a+3} \cdot m - b_{a+1,3} - 2^{a+3} + 1)} \equiv 3 \bmod 4. \tag{3}$$

Observe that the numerator and denominator of the left hand side of (3) have 2^{a+2} consecutive odd terms. Thus, the left hand side of (3) is congruent to

$$\frac{(2^{a+3} \cdot m - 2^{a+1} + b_{a+1,3} + 2^{a+3})(2^{a+3} \cdot m - 2^{a+1} + b_{a+1,3} + 2^{a+3} - 2) \cdots (2^{a+3} \cdot m - 2^{a+1} + b_{a+1,3})}{(2^{a+3} \cdot m - b_{a+1,3})(2^{a+3} \cdot m - b_{a+1,3} - 2) \cdots (2^{a+3} \cdot m - b_{a+1,3} - 2^{a+3})}$$

modulo 4. Now factor a 2 out of each term to obtain

$$\frac{(2^{a+2} \cdot m - 2^a + b_{a+1,3}/2 + 2^{a+2}) \cdots (2^{a+2} \cdot m - 2^a + b_{a+1,3}/2)}{(2^{a+2} \cdot m - b_{a+1,3}/2) \cdots (2^{a+2} \cdot m - b_{a+1,3}/2 - 2^{a+2})}. \tag{4}$$

Since $b_{a+1,3}/2 = b_{a,3}$, then (4) is the same number as the left hand side of (2) and so, by our induction hypothesis, (3) is congruent to 3 mod 4, which is what we wanted to prove. This takes care of the first step of the double induction.

Suppose that

$$\frac{(2^{a+i-1} \cdot m - 2^a + b_{a,i} + 2^{a+i-1}) \cdots (2^{a+i-1} \cdot m - 2^a + b_{a,i} + 1)}{(2^{a+i-1} \cdot m - b_{a,i}) \cdots (2^{a+i-1} \cdot m - b_{a,i} - 2^{a+i-1} + 1)} \equiv 3 \bmod 4. \tag{5}$$

is true for some $i \geq 3$ Consider the case $i + 1$, i.e.

$$\frac{(2^{a+i} \cdot m - 2^a + b_{a,i+1} + 2^{a+i}) \cdots (2^{a+i} \cdot m - 2^a + b_{a,i+1} + 1)}{(2^{a+i} \cdot m - b_{a,i+1}) \cdots (2^{a+i} \cdot m - b_{a,i+1} - 2^{a+i} + 1)}.$$

Note that there are 2^{a+i} consecutive integers in the numerator and in the denominator. Thus, we have 2^{a+i-1} consecutive odd numbers in the numerator and in the denominator. Since $a + i - 1 \geq 2$, then we have that these terms will be congruent to 1 mod 8. After this cancellation, we are left with the even terms:

$$\frac{(2^{a+i} \cdot m - 2^a + b_{a,i+1} + 2^{a+i}) \cdots (2^{a+i} \cdot m - 2^a + b_{a,i+1} + 2)}{(2^{a+i} \cdot m - b_{a,i+1}) \cdots (2^{a+i} \cdot m - b_{a,i+1} - 2^{a+i} + 2)}. \tag{6}$$

Observe that if $a \geq 2$, then, after taking this expression mod 4, we have

$$\frac{(2^{a+i} \cdot m + b_{a,i+1} + 2^{a+i}) \cdots (2^{a+i} \cdot m + b_{a,i+1} + 2)}{(2^{a+i} \cdot m - b_{a,i+1}) \cdots (2^{a+i} \cdot m - b_{a,i+1} - 2^{a+i} + 2)} \bmod 4. \tag{7}$$

Now factor a 2 out of each term to obtain

$$\frac{(2^{a+i-1} \cdot m + b_{a,i} + 2^{a+i-1}) \cdots (2^{a+i-1} \cdot m + b_{a,i} + 1)}{(2^{a+i-1} \cdot m - b_{a,i}) \cdots (2^{a+i-1} \cdot m - b_{a,i} - 2^{a+i-1} + 1)} \bmod 4, \tag{8}$$

and this expression is congruent to 3 modulo 4 by induction. If $a = 1$, then observe that (5) gets transformed to

$$\frac{(2^i \cdot m + b_{1,i} + 2^i - 2)(2^i \cdot m + b_{1,i} + 2^i - 1) \cdots (2^i \cdot m + b_{1,i} - 1)}{(2^i \cdot m - b_{1,i}) \cdots (2^i \cdot m - b_{1,i} - 2^i + 1)} \equiv 3 \bmod 4. \tag{9}$$

while (6) gets transformed to

$$\frac{(2^{i+1} \cdot m + b_{1,i+1} + 2^{i+1} - 2) \cdots (2^{i+1} \cdot m + b_{1,i+1})}{(2^{i+1} \cdot m - b_{1,i+1}) \cdots (2^{i+1} \cdot m - b_{1,i+1} - 2^{i+1} + 2)}. \tag{10}$$

Factor a 2 out of each term to obtain

$$\frac{(2^i \cdot m + b_{1,i} + 2^i - 1) \cdots (2^i \cdot m + b_{1,i})}{(2^i \cdot m - b_{1,i}) \cdots (2^i \cdot m - b_{1,i} - 2^i + 1)}. \tag{11}$$

Since $2^i - 1 \equiv -1 \bmod 8$, then (11) is congruent to (9) modulo 4, and thus, congruent to 3 mod 4 by induction. This concludes the proof. □

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