A CLASS OF LOGARITHMIC INTEGRALS

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Abstract. We present a systematic study of integrals of the form

$$I_Q = \int_0^1 Q(x) \log\log \frac{1}{x}\, dx,$$

where Q is a rational function.

1. Introduction

The classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik [8] contains very few evaluations of the form

$$I_Q := \int_0^1 Q(x) \log\log 1/x\, dx, \tag{1.1}$$

where Q is a rational function. The example

$$\int_0^1 \frac{1}{x^2+1} \log\log 1/x\, dx = \frac{\pi}{2} \log\left(\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right), \tag{1.2}$$

written in its trigonometric version

$$\int_0^1 \frac{1}{x^2+1} \log\log 1/x\, dx = \int_{\pi/4}^{\pi/2} \log\log \tan x\, dx, \tag{1.3}$$

is the subject of Vardi's remarkable paper [13]. This example appears as 4.229.7 in [8] and also in the equivalent form

$$\int_0^\infty \frac{\log x\, dx}{\cosh x} = \pi \log\left(\frac{\sqrt{2\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right), \tag{1.4}$$

as 4.371.1 in the same table.

We present here a systematic study of the logarithmic integrals (1.1). Throughout the paper we indicate whether Mathematica 6.0 is capable of evaluating the integrals considered. For example, a direct symbolic evaluation gives (1.2) as

$$\int_0^1 \frac{1}{x^2+1} \log\log 1/x\, dx = \frac{\pi}{4} \log\left(\frac{4\pi^3}{\Gamma(\frac{1}{4})^4} \right). \tag{1.5}$$

The reader should be aware that the question of whether a definite integral is computable by a symbolic language depends on the form in which the integrand is expressed. For instance, Mathematica 6.0 is unable to evaluate the trigonometric version of (1.2) given as the right-hand side of (1.3).

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The idea exploited here, introduced by I. Vardi in [13], is to associate to each function Q a gamma factor

$$\Gamma_Q(s) := \int_0^1 Q(x) \left(\log \frac{1}{x} \right)^{s-1} dx, \tag{1.6}$$

so that, the integral (1.1) is given by

$$I_Q = \Gamma_Q'(1). \tag{1.7}$$

An explicit evaluation of I_Q is achieved in the case where $Q(x)$ is analytic at $x = 0$. Starting with the expansion

$$Q(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{1.8}$$

we associate an L-series

$$L_Q(s) := \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^s}. \tag{1.9}$$

The integral (1.1) is now evaluated as

$$I_Q = -\gamma L_Q(1) + L_Q'(1), \tag{1.10}$$

where γ is the *Euler-Mascheroni* constant. The interesting story of this fundamental constant can be found in [10]. The identity (1.10) is essentially Vardi's method for the evaluation of (1.2). Naturally, to obtain an *explicit* evaluation of I_Q, one needs to express $L_Q(1)$ and $L_Q'(1)$ in terms of special functions. We employ here the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1, \tag{1.11}$$

and its alternating form

$$\zeta_a(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = -(1 - 2^{1-s})\zeta(s), \quad s > 0. \tag{1.12}$$

The relations

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}B_{2n}}{2(2n)!}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{1.13}$$

$$\zeta(1-n) = \frac{(-1)^{n+1}B_n}{n}, \quad n \in \mathbb{N}, \tag{1.14}$$

$$\zeta'(-2n) = (-1)^n \frac{(2n)!\, \zeta(2n+1)}{2(2\pi)^{2n}}, \quad n \in \mathbb{N}, \tag{1.15}$$

$$\zeta'(0) = -\log \sqrt{2\pi}, \tag{1.16}$$

where B_n are the Bernoulli numbers, will be used to simplify the integrals discussed below.

A second function that is used in the evaluations described here is the *polylogarithm function* defined by

$$\text{PolyLog}[c, x] := \sum_{n=1}^{\infty} \frac{x^n}{n^c} \tag{1.17}$$

and its derivative

$$\text{PolyLog}^{(1,0)}[c,x] := \frac{d}{dc}\text{PolyLog}[c,x] = -\sum_{n=1}^{\infty} \frac{\log n}{n^c} x^n. \tag{1.18}$$

A complete description of I_Q is determined here in the case where $Q(x)$ is a rational function. The method of partial fractions shows that it suffices to consider three types of integrals:

the first type is

$$P_j := \int_0^1 x^j \log\log 1/x \, dx, \tag{1.19}$$

that gives the polynomial part of Q,

the second type is

$$R_{m,j}(a) := \int_0^1 \frac{x^j \, \log\log 1/x}{(x+a)^{m+1}} \, dx, \tag{1.20}$$

that treats the real poles of Q, and

the third type is

$$C_{m,j}(a,b) := \int_0^1 \frac{x^j \, \log\log 1/x}{(x^2+ax+b)^{m+1}} \, dx, \tag{1.21}$$

with $a^2 - 4b < 0$. This last case deals with the non-real poles of Q.

Integrals of first type. These are simple. They are evaluated in (2.10) as

$$P_j := \int_0^1 x^j \log\log 1/x \, dx = -\frac{\gamma + \log(j+1)}{j+1}.$$

Integrals of second type. The special case $R_{m,0}(1)$ is evaluated first. We introduce the polynomial

$$T_m(x) := \sum_{j=0}^{m} (-1)^j A_{m+1,j+1} x^j, \tag{1.22}$$

with $A_{m,j}$ the Eulerian numbers given in (4.6). Then the integral

$$E_m := \int_0^1 \frac{T_{m-1}(x) \, \log\log 1/x}{(x+1)^{m+1}} \, dx \tag{1.23}$$

is evaluated, for $m > 1$, as

$$E_m = (1-2^m)\zeta'(1-m) + (-1)^m \left(\gamma(2^m-1) + 2^m \log 2\right) \frac{B_m}{m}. \tag{1.24}$$

The sequence E_m is then used to produce a recurrence for $R_{m,0}(1)$. The initial condition

$$R_{0,0}(1) = -\frac{1}{2}\log^2 2 \tag{1.25}$$

is given in Example 4.1. Then, the values of $R_{m,0}(1)$ are obtained from

$$R_{m,0}(1) = \frac{E_m}{b_0(m)} - \sum_{k=1}^{m-1} \frac{b_k(m)}{b_0(m)} R_{m-k,0}(1), \tag{1.26}$$

where

$$b_k(m) = (-1)^k \sum_{j=0}^{m-1} \binom{j}{k} A_{m,j+1}. \tag{1.27}$$

This is described in Corollary 4.7.

The evaluation of $R_{m,0}(a)$, for $a \neq 1$, appears in Proposition 5.1 and Corollary 5.3: the value of $R_{0,0}(a)$ is given by

$$R_{0,0}(a) := \int_0^1 \frac{\log\log 1/x \, dx}{x+a} = -\gamma \log(1+1/a) - \text{PolyLog}^{(1,0)}[1, -1/a], \tag{1.28}$$

and, for $m > 0$, we have

$$\begin{aligned} R_{m,0}(a) &= -\frac{\gamma}{a^m(1+a)m} - \frac{\gamma}{a^{m+1}m!} \sum_{j=2}^{m} \frac{S_1(m,j)T_{j-2}(1/a)}{(1+1/a)^j} \\ &\quad - \frac{1}{a^m m!} \sum_{j=1}^{m} S_1(m,j) \text{PolyLog}^{(1,0)}[1-j, -1/a]. \end{aligned}$$

Here $S_1(m,j)$ are the (signless) Stirling numbers of the first kind defined by the expansion

$$(t)_m = \sum_{j=1}^{m} S_1(m,j) t^j, \tag{1.29}$$

where $(t)_m = t(t+1)(t+2)\cdots(t+m-1)$ is the Pochhammer symbol. The function $\text{PolyLog}^{(1,0)}[c,x]$ is defined in (1.18).

For $j > 0$, the value of $R_{m,j}(a)$ is now obtained from the recursion in Theorem 6.1, written here as

$$R_{m,0}(a) = \sum_{j=0}^{r} \alpha_{j,r}(a) R_{m-r+j,j}(a), \tag{1.30}$$

where

$$\alpha_{j,r}(a) := (-1)^j \binom{r}{j} a^{-r}. \tag{1.31}$$

This can be used for increasing values of the free parameter r, to obtain analytic expressions for $R_{m,j}(a)$. For instance, $r = 1$ gives

$$R_{m,0}(a) = \alpha_{0,1}(a)R_{m-1,0}(a) + \alpha_{1,1}(a)R_{m,1}(a), \tag{1.32}$$

that determines $R_{m,1}(a)$ in terms of $R_{m,0}(a)$ and $R_{m-1,0}(a)$, that were previously computed. The value $r = 2$ gives

$$R_{m,0}(a) = \alpha_{0,2}(a)R_{m-2,0}(a) + \alpha_{1,2}(a)R_{m-1,1}(a) + \alpha_{2,2}(a)R_{m,2}(a), \tag{1.33}$$

that determines $R_{m,2}(a)$ in terms of previously computed integrals. This procedure determines all the integrals $R_{m,j}(a)$.

Integrals of third type. These are integrals where the corresponding quadratic factor has non-real zeros. The expression

$$x^2 + ax + b = (x - c)(x - \bar{c}) = x^2 - 2rx\cos\theta + r^2 \tag{1.34}$$

is used to define

$$D_{m,j}(r,\theta) = \int_0^1 \frac{x^j \, \log\log 1/x}{(x^2 - 2rx\cos\theta + r^2)^{m+1}}\, dx. \tag{1.35}$$

Naturally $C_{m,j}(a,b) = D_{m,j}(r,\theta)$, we are simply emphasizing the polar representation of the poles.

The value $D_{0,0}(r,\theta)$ is computed first. Theorem 7.1 treats the case $r = 1$, with the value

$$D_{0,0}(1,\theta) = \frac{\pi}{2\sin\theta}\left[(1-\theta/\pi)\log 2\pi + \log\left(\frac{\Gamma(1-\theta/2\pi)}{\Gamma(\theta/2\pi)}\right)\right].$$

The case $r \neq 1$ is given in Theorem 7.2 as

$$\begin{aligned} D_{0,0}(r,\theta) &= -\frac{\gamma}{r\sin\theta}\tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right) \\ &\quad + \frac{1}{2ri\sin\theta}\left(\text{PolyLog}^{(1,0)}[1, e^{i\theta}/r] - \text{PolyLog}^{(1,0)}[1, e^{-i\theta}/r]\right). \end{aligned}$$

The next step is to compute $D_{0,1}(r,\theta)$. This is described at the end of Section 7. The result is expressed in terms of the *Lerch zeta function*

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}, \tag{1.36}$$

as

$$\begin{aligned} D_{0,1}(r,\theta) &= -\frac{\gamma}{2}\log\left(\frac{r^2 - 2r\cos\theta + 1}{r^2}\right) \\ &\quad - \gamma\cot\theta\,\tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right) \\ &\quad + \frac{1}{2ri\sin\theta}\left[\Phi^{(0,1,0)}\left(\frac{e^{i\theta}}{r},1,1\right) - \Phi^{(0,1,0)}\left(\frac{e^{-i\theta}}{r},1,1\right)\right]. \end{aligned}$$

The reader will find in [11] information about this function.

The values of $D_{m,j}(r,\theta)$ for $m,\, j > 0$, are determined by the recurrences

$$D_{m,j}(r,\theta) = -\frac{1}{2rm\sin\theta}\frac{\partial}{\partial\theta}D_{m-1,j-1}(r,\theta), \tag{1.37}$$

and

$$D_{m,j}(r,\theta) = \frac{1}{2m\cos\theta}\left(\frac{\partial}{\partial r}D_{m-1,j-1}(r,\theta) + 2rmD_{m,j-1}(r,\theta)\right). \tag{1.38}$$

These follow directly from the definition of $D_{m,j}(r,\theta)$. Details are given in Section 7.

Comment. Integration by parts, shows that the integrals I_Q in (1.1) include those of the form

$$J_Q = \int_0^1 \frac{Q(x)\,dx}{\log x}. \tag{1.39}$$

This class was originally studied by V. Adamchik [1]. They were considered by Baxter, Tempereley and Ashley [3] in their work on the so-called Potts model for the triangular lattice. In that model, the generating function has the form

$$P_3(t) := 3\int_0^\infty \frac{\sinh((\pi-t)x)\,\sinh\left(\frac{2tx}{3}\right)}{x\sinh(\pi x)\cosh(tx)}\,dx. \tag{1.40}$$

V. Adamchik determined analytic expressions for (1.39) in the case where the denominator of Q is a cyclotomic polynomial. The expressions involve derivatives of the Hurwitz zeta function

$$\zeta(z,q) := \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}. \tag{1.41}$$

Zhang Nan-Hue and K. Williams [14], [15] used the Hurwitz zeta function to evaluate definite integrals similar to the class considered here. Examples of integrals that involve $\zeta(z,q)$ in the integrand are given in [5], [6] and [7].

2. The main tool

The evaluation of the integral

$$I_Q := \int_0^1 Q(x)\log\log 1/x\,dx, \tag{2.1}$$

for a general function $Q(x)$, is achieved by introducing the function

$$\Gamma_Q(s) := \int_0^1 Q(x)\,(\log 1/x)^{s-1}\;dx. \tag{2.2}$$

The next result is elementary.

Lemma 2.1. The integral I_Q is given by

$$I_Q = \Gamma_Q'(1). \tag{2.3}$$

Example 2.1. The simplest case is $Q(x)\equiv 1$. Here we obtain

$$\int_0^1 \log\log 1/x\,dx = \Gamma'(1), \tag{2.4}$$

where

$$\Gamma(s) = \int_0^1 \left(\log\frac{1}{x}\right)^{s-1} dx = \int_0^\infty t^{s-1}e^{-t}\,dt, \tag{2.5}$$

is the classical *gamma function*. The reader will find in [4] the identity

$$\Gamma'(1) = -\gamma, \tag{2.6}$$

where γ is the *Euler-Mascheroni* constant. This example appears as 4.229.1 in [8].

Example 2.2. Consider now the case $Q(x)=x^a$, for $a\in\mathbb{R}$. Observe that

$$\Gamma_{x^a}(s) = \int_0^\infty e^{-(a+1)t}t^{s-1}\,dt = \frac{\Gamma(s)}{(a+1)^s}, \text{ for } a>-1 \text{ and } s>0. \tag{2.7}$$

Differentiate with respect to s at $s=1$ to produce

$$\int_0^1 x^a \log\log 1/x\,dx = -\frac{\gamma+\log(a+1)}{a+1}. \tag{2.8}$$

Differentiating (2.7) n times with respect to s, yields

$$\int_0^1 x^a \log^n x \log\log 1/x\,dx = \frac{(-1)^{n+1}\,n!}{(1+a)^{n+1}}\left(\log(1+a)+\gamma-H_n\right), \tag{2.9}$$

where H_n is the n-th harmonic number. Mathematica 6.0 is unable to evaluate (2.7) if both a and n are entered as parameters. The same holds for (2.9).

Note 2.2. The expression (2.8), with $a = m \in \mathbb{N}$, provides the evaluation of the integral P_m in (1.19):

$$P_m := \int_0^1 x^m \log\log 1/x\,dx = -\frac{\gamma+\log(m+1)}{m+1}. \tag{2.10}$$

This appears as 4.325.8 in [8].

3. The case where Q is analytic at $x = 0$

In this section we consider the evaluation of the integral

$$I_Q := \int_0^1 Q(x)\log\log 1/x\,dx, \tag{3.1}$$

where Q is admits an expansion

$$Q(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{3.2}$$

The expression for I_Q is expressed in terms of the associated L-function defined by

$$L_Q(s) := \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^s}. \tag{3.3}$$

The idea for the next lemma comes from [13].

Lemma 3.1. The function Γ_Q satisfies $\Gamma_Q(s) = \Gamma(s)L_Q(s)$.

Proof. The linearity of $\Gamma_Q(s)$ in the Q-argument shows that

$$\Gamma_Q(s) = \sum_{n=0}^{\infty} a_n \Gamma_{x^n}(s). \tag{3.4}$$

The result now follows from the value of $\Gamma_{x^n}(s)$ in (2.7). □

Theorem 3.2. Assume Q is given by (3.2). Then

$$I_Q := \int_0^1 Q(x)\log\log 1/x\,dx = -\gamma L_Q(1) + L_Q'(1). \tag{3.5}$$

Proof. Differentiate the expression for Γ_Q in the previous lemma and use the result of Lemma 2.1. □

The theorem reduces the evaluation of I_Q to the evaluation of $L_Q(1)$ and $L_Q'(1)$. The first series of examples come from prescribing the coefficients a_n of $Q(x)$ so that the L_Q function is relatively simple.

Example 3.1. Choose $a_n = 1/(n+1)$. Then

$$Q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1} = -\frac{\log(1-x)}{x}. \tag{3.6}$$

Then

$$L_Q(s) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{s+1}} = \zeta(s+1), \tag{3.7}$$

where $\zeta(s)$ is the classical Riemann zeta function. Theorem 3.2 gives

$$\int_0^1 \frac{\log(1-x)}{x} \log\log 1/x \, dx = \frac{\gamma\pi^2}{6} - \zeta'(2). \tag{3.8}$$

Mathematica 6.0 is unable to check this directly, but the change of variables $x = e^{-t}$ transforms (3.8) to

$$\int_0^{\infty} \log t \log(1-e^{-t}) \, dt = \frac{\gamma\pi^2}{6} - \zeta'(2). \tag{3.9}$$

This is computable by Mathematica 6.0.

The constant

$$\zeta'(2) = -\sum_{n=1}^{\infty} \frac{\log n}{n^2} \tag{3.10}$$

can be expressed in terms of the *Glaisher constant*

$$\log A := \frac{1}{12} - \zeta'(-1), \tag{3.11}$$

by

$$\zeta'(2) = \frac{\pi^2}{6} \left(\gamma + \log(2\pi) - 12 \log A \right). \tag{3.12}$$

This gives

$$\int_0^1 \frac{\log(1-x)}{x} \log\log 1/x \, dx = \frac{\pi^2}{6} \left(12 \log A - \log 2\pi \right) \tag{3.13}$$

as an alternative form for (3.8).

Example 3.2. We now consider the alternating version of Example 3.1 and choose $a_n = (-1)^n/(n+1)$. In this case

$$Q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = \frac{\log(1+x)}{x}, \tag{3.14}$$

and the corresponding L-series is

$$L_Q(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{s+1}} = (1-2^{-s})\zeta(s+1). \tag{3.15}$$

Theorem 3.2 and the evaluations

$$L_Q(1) = \frac{\pi^2}{12} \text{ and } L_Q'(1) = \frac{\pi^2 \log 2}{12} + \frac{1}{2}\zeta'(2), \tag{3.16}$$

now yield

$$\int_0^1 \frac{\log(1+x)}{x} \log\log 1/x \, dx = \frac{\pi^2}{12}(\log 2 - \gamma) + \frac{1}{2}\zeta'(2). \tag{3.17}$$

This can also be expressed as

$$\int_0^1 \frac{\log(1+x)}{x} \log\log 1/x \, dx = \frac{\pi^2}{12}(\log 4\pi - 12\log A). \tag{3.18}$$

As in the previous example, Mathematica 6.0 is unable to produce this evaluation, but it succeeds with the alternate version

$$\int_0^\infty \log t \log(1+e^{-t}) \, dt = \frac{\pi^2}{12}(\log 2 - \gamma) + \frac{1}{2}\zeta'(2). \tag{3.19}$$

Example 3.3. Adding the results of the first two examples yields

$$\int_0^1 \frac{\log(1-x^2)}{x} \log\log 1/x \, dx = \frac{\pi^2}{12}(\log 2 + \gamma) - \frac{1}{2}\zeta'(2). \tag{3.20}$$

Their difference produces

$$\int_0^\infty \log t \log \tanh t \, dt = \frac{\gamma\pi^2}{8} - \frac{3}{4}\zeta'(2) + \frac{\pi^2 \log 2}{12}. \tag{3.21}$$

This cannot be evaluated symbolically.

Example 3.4. This example generalizes Example 3.1. The integrand involves the *polylogarithm* function defined in (1.17). The choice $a_n = 1/(n+1)^c$ produces the function

$$Q(x) = \sum_{n=0}^\infty \frac{x^n}{(n+1)^c} = \frac{1}{x}\text{PolyLog}[c, x], \tag{3.22}$$

and the corresponding L-function is

$$L_Q(s) = \sum_{n=0}^\infty \frac{1}{(n+1)^{s+c}} = \zeta(s+c). \tag{3.23}$$

Then (3.5) gives

$$\int_0^1 \frac{\text{PolyLog}[c, x]}{x} \log\log 1/x \, dx = -\gamma\zeta(c+1) + \zeta'(c+1). \tag{3.24}$$

Example 3.5. Choosing now $a_n = (-1)^n/(n+1)^c$ gives

$$Q(x) = \sum_{n=0}^\infty \frac{(-1)^n x^n}{(n+1)^c} = -\frac{1}{x}\text{PolyLog}[c, -x] \tag{3.25}$$

and

$$L_Q(s) = \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^{s+c}} = (1 - 2^{1-s-c})\zeta(s+c). \tag{3.26}$$

We conclude that

$$\int_0^1 \frac{\text{PolyLog}[c, -x]}{x} \log\log 1/x \, dx = \left(\gamma(1-2^{-c}) - 2^{-c}\log 2\right)\zeta(c+1) - (1-2^{-c})\zeta'(c+1).$$

In the special case where c is a negative integer, the function Q reduces to a rational function. Details are provided in section 4.

Example 3.6. Interesting integrands can be produced when a_n that are periodic sequences. For example, the choice

$$a_n = \frac{1}{n}\cos\left(\frac{2\pi n}{3}\right) \tag{3.27}$$

gives the function

$$Q(x) = \sum_{n=1}^{\infty} \frac{1}{n}\cos\left(\frac{2\pi n}{3}\right) x^{n-1} = -\frac{\log(1+x+x^2)}{2x}, \tag{3.28}$$

and the corresponding L-function

$$L_Q(s) = \sum_{n=1}^{\infty} \frac{\cos(\frac{2\pi n}{3})}{n^{s+1}} = -\frac{1-3^{-s}}{2}\zeta(s+1). \tag{3.29}$$

Then (3.5) gives

$$\int_0^1 \frac{\log(1+x+x^2)}{x}\log\log 1/x\,dx = -\frac{\gamma\pi^2}{9} + \frac{1}{18}\pi^2\log 3 + \frac{2}{3}\zeta'(2). \tag{3.30}$$

Example 3.7. This example presents a second periodic sequence. The choice

$$a_n = \frac{1}{n}\cos\left(\frac{2\pi n}{5}\right) \tag{3.31}$$

gives the function

$$Q(x) = \sum_{n=1}^{\infty} \frac{1}{n}\cos\left(\frac{2\pi n}{5}\right) x^{n-1} = -\frac{\log(1-\varphi x+x^2)}{2x}, \tag{3.32}$$

where $\varphi = (\sqrt{5}-1)/2$. The corresponding L-function is

$$L_Q(s) = \sum_{n=1}^{\infty} \frac{\cos(\frac{2\pi n}{5})}{n^{s+1}} = \frac{1}{2}\left(\text{PolyLog}[s+1, e^{-2\pi i/5}] + \text{PolyLog}[s+1, e^{2\pi i/5}]\right).$$

The values $L_Q(1) = \pi^2/150$ and

$$L_Q'(1) = -\sum_{n=1}^{\infty} \frac{\log n}{n^2}\cos\left(\frac{2\pi n}{5}\right), \tag{3.33}$$

and Theorem 3.2 give the identity

$$\int_0^1 \frac{\log(1-\varphi x+x^2)}{2x}\log\log 1/x\,dx = \frac{\gamma\pi^2}{150} + \sum_{n=1}^{\infty} \frac{\log n}{n^2}\cos\left(\frac{2\pi n}{5}\right). \tag{3.34}$$

Example 3.8. The result of Theorem 3.2 reduces the evaluation of a certain class of integrals to the evaluation of the corresponding L-functions. Many natural choices of the function Q lead to series that the authors are unable to evaluate. For example, $Q(x) = e^x$ produces the identity

$$\int_0^1 e^x \log\log 1/x\,dx = -\gamma(e-1) - \sum_{n=1}^{\infty} \frac{\log n}{n!}, \tag{3.35}$$

and we have been unable to procude an analytic expression for the series above. The same is true for the series in (3.34).

4. Evaluation of integrals with real poles. An expression for $R_{m,0}(1)$

We now turn to the evaluation of the integrals

$$R_{m,j}(a) = \int_0^1 \frac{x^j \,\log\log 1/x}{(x+a)^{m+1}}\, dx. \tag{4.1}$$

The method of partial fractions can then be used to produce explicit formulas for integrals of the type

$$I_Q = \int_0^1 Q(x)\,\log\log 1/x\, dx \tag{4.2}$$

where Q is a rational function with only real poles.

In this section we introduce a special family of polynomials $T_m(x)$ and produce an explicit analytic expression for

$$E_m := \int_0^1 \frac{T_{m-1}(x)\,\log\log 1/x}{(x+1)^{m+1}}\, dx. \tag{4.3}$$

These are then employed to evaluate $R_{m,0}(1)$.

Definition 4.1. The *Eulerian polynomials* A_m are defined by the generating function

$$\frac{1-x}{1-x\exp[t(1-x)]} = \sum_{m=0}^{\infty} A_m(x)\frac{t^m}{m!}. \tag{4.4}$$

Note 4.2. The Eulerian polynomials appear in many combinatorial problems. The coefficients $A_{m,j}$ in

$$A_m(x) = \sum_{j=1}^{m} A_{m,j}x^j \tag{4.5}$$

are the *Eulerian numbers*. They count the number of permutations of $\{1,\, 2,\, \cdots, n\}$ which show exactly j increases between adjacent elements, the first element always being counted as a jump. The numbers $A_{m,j}$ have an explicit formula

$$A_{m,j} = \sum_{k=0}^{j} (-1)^k \binom{m+1}{k} (j-k)^m, \tag{4.6}$$

and a recurrence relation

$$A_{m,j} = jA_{m-1,j} + (m-j+1)A_{m-1,j-1} \tag{4.7}$$

that follows from

$$A_{m+1}(x) = x(1-x)\frac{d}{dx}A_m(x) + (m+1)xA_m(x), \tag{4.8}$$

with $A_0(x) = 1$. The recurrence (4.8) follows directly from (4.4) and it immediatly implies that $A_m(x)$ is a polynomial of degree m. The first few are

$$\begin{aligned} A_0(x) &= 1, \\ A_1(x) &= x, \\ A_2(x) &= x^2 + x, \\ A_3(x) &= x^3 + 4x^2 + x, \\ A_4(x) &= x^4 + 11x^3 + 11x^2 + x. \end{aligned}$$

More information about these polynomials can be found in [9].

We now present the relation between Eulerian polynomials and the polylogarithm function.

Lemma 4.3. Let $m \in \mathbb{N}$. The polynomial $A_m(x)$ satisfies

$$\text{PolyLog}[-m, -x] = \frac{A_m(-x)}{(x+1)^{m+1}}. \tag{4.9}$$

Proof. The identity

$$\text{PolyLog}[-m, -x] = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1)^m x^{n+1}, \tag{4.10}$$

shows that

$$\text{PolyLog}[-m, -x] = \vartheta^{(m)} \left(\frac{1}{1+x} \right), \tag{4.11}$$

where $\vartheta = x \frac{d}{dx}$. The claim now follows by using (4.8) and an elementary induction. □

We now employ Eulerian polynomials to evaluate an auxiliary family of integrals.

Proposition 4.4. Let $m \in \mathbb{N}$. Define

$$T_m(x) := -\frac{A_{m+1}(-x)}{x} = \sum_{j=0}^{m} (-1)^j A_{m+1,j+1} x^j, \tag{4.12}$$

and

$$E_m := \int_0^1 \frac{T_{m-1}(x) \log\log 1/x}{(x+1)^{m+1}} \, dx. \tag{4.13}$$

Then

$$E_m = (1 - 2^m)\zeta'(1-m) + \left(\gamma(2^m - 1) + 2^m \log 2\right) \zeta(1-m), \quad \text{for } m \geq 1. \tag{4.14}$$

Proof. This is a special case of Example 3.4. □

Note 4.5. The relation (1.14) gives

$$E_m = (1-2^m)\zeta'(1-m) + (-1)^{m+1} \left(\gamma(2^m - 1) + 2^m \log 2\right) \frac{B_m}{m}, \quad \text{for } m \geq 1. \tag{4.15}$$

For example,

$$
\begin{aligned}
E_1 &= -\frac{\gamma}{2} + \frac{\log \pi}{2} - \frac{\log 2}{2}, \\
E_2 &= -\frac{1}{4} - \frac{\gamma}{4} - \frac{\log 2}{3} + 3\log A,
\end{aligned} \tag{4.16}
$$

where A is the Glaisher constant defined in (3.11).

The expression for E_m in (4.15) is now used to provide a recurrence for the integrals $R_{m,0}(1)$, where $R_{m,j}(a)$ is defined in (1.20). We begin with an example that will provide an initial condition for the recurrence in Corollary 4.7.

Example 4.1. The integral $R_{0,0}(1)$ is given by

$$
R_{0,0}(1) = \int_0^1 \frac{\log\log 1/x \, dx}{1+x} = -\frac{\log^2 2}{2}. \tag{4.17}
$$

The choice $a_n = (-1)^n$ in Theorem 3.2 produces $Q(x) = 1/(1+x)$ and $L_Q(s) = (1-2^{1-s})\zeta(s)$. Passing to the limit as $s \to 1$ and using $R_{0,0}(1) = -\gamma L_Q(1) + L_Q'(1)$ gives the result.

Theorem 4.6. The integrals E_m in (4.15) satisfy

$$
E_m = \sum_{k=0}^{m-1} b_k(m) R_{m-k,0}(1), \tag{4.18}
$$

where

$$
b_k(m) = (-1)^k \sum_{j=k}^{m-1} \binom{j}{k} A_{m,j+1} \tag{4.19}
$$

and $A_{m,j}$ are the Eulerian numbers given in (4.6).

Proof. In the expression

$$
E_m = \int_0^1 \frac{T_{m-1}(x)\, \log\log 1/x}{(x+1)^{m+1}}\, dx \tag{4.20}
$$

use (4.12) to obtain

$$
E_m = \sum_{j=0}^{m-1} (-1)^j A_{m,j+1} \int_0^1 \frac{x^j\, \log\log 1/x}{(x+1)^{m+1}}\, dx. \tag{4.21}
$$

Now write $x = (x+1) - 1$, expand the resulting binomial and reverse the order of summation to obtain the result. $\square$

Corollary 4.7. The integrals $R_{m,0}(1)$ satisfy the recurrence

$$
R_{1,0}(1) = E_1 \tag{4.22}
$$

$$
R_{m,0}(1) = \frac{E_m}{b_0(m)} - \sum_{k=1}^{m-1} \frac{b_k(m)}{b_0(m)} R_{m-k,0}(1). \tag{4.23}
$$

Proof. First observe that $b_0(m) \neq 0$. Indeed,

$$
b_0(m) = \sum_{j=1}^{m} A_{m,j} = A_m(1). \tag{4.24}
$$

Using the recurrrence (4.8) we conclude that $A_{m+1}(1) = (m+1)A_m(1)$. Therefore $b_0(m) = A_m(1) = m!$. □

Example 4.2. The previous result provides the values

$$\begin{aligned} R_{1,0}(1) &= \frac{1}{2}\left(-\gamma + \log \pi - \log 2\right), \\ R_{2,0}(1) &= \frac{1}{24}\left(-3 - 9\gamma - 10\log 2 + 36\log A + 6\log \pi\right) \\ R_{3,0}(1) &= \frac{1}{24}\left(-3 - 7\gamma - 8\log 2 + 36\log A + 4\log \pi + 7\zeta(3)/\pi^2\right). \end{aligned}$$

These integrals are computable using Mathematica 6.0.

5. An expression for $R_{m,0}(a)$

In this section we present an analytic expression for

$$R_{m,0}(a) := \int_0^1 \frac{\log\log 1/x}{(x+a)^{m+1}}\, dx. \tag{5.1}$$

The result is given in terms of the polylogarithm function defined in (1.17) and the derivative $\text{PolyLog}^{(1,0)}[c,x]$ defined in (1.18). The evaluation employs the expression for $R_{m,0}(1)$ given the previous section.

Proposition 5.1. The integral $R_{0,0}(a)$ is given by

$$R_{0,0}(a) := \int_0^1 \frac{\log\log 1/x\, dx}{x+a} = -\gamma\log(1+1/a) - \text{PolyLog}^{(1,0)}[1,-1/a]. \tag{5.2}$$

Proof. The expansion

$$Q(x) = \frac{1}{x+a} = \frac{1}{a}\sum_{n=0}^{\infty}(-1)^n a^{-n} x^n, \tag{5.3}$$

produces the L-function

$$L_Q(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}(n+1)^s} = -\text{PolyLog}[s,-1/a]. \tag{5.4}$$

Theorem 3.2 gives the result. □

The evaluation of $R_{m,0}(a)$ for $m \geq 1$ employs the *signless Stirling numbers of the first kind* $S_1(m,j)$ defined by the expansion

$$(t)_m = \sum_{j=1}^{m} S_1(m,j)t^j, \tag{5.5}$$

where $(t)_m = t(t+1)(t+2)\cdots(t+m-1)$ is the Pochhammer symbol.

Theorem 5.2. Let $m \in \mathbb{N}$ and $a > 0$. The L-function associated to $Q(x) = 1/(x+a)^{m+1}$ is

$$L_Q(s) = -\frac{1}{a^m m!}\sum_{j=1}^{m} S_1(m,j)\text{PolyLog}[s-j,-1/a]. \tag{5.6}$$

Proof. The identity

$$\binom{-\beta}{k} = \frac{(-1)^k(\beta)_k}{k!} \tag{5.7}$$

is used in the expansion

$$\frac{1}{(x+a)^\beta} = a^{-\beta}(1+x/a)^{-\beta} = a^{-\beta}\sum_{k=0}^{\infty}\binom{-\beta}{k}a^{-k}x^k \tag{5.8}$$

to produce

$$\frac{1}{(x+a)^\beta} = a^{-\beta}\sum_{k=0}^{\infty}\frac{(-1/a)^k(\beta)_k}{k!(k+1)^s}. \tag{5.9}$$

Now choose $\beta = m+1$ and use the elementary identity

$$\frac{(m+1)_k}{k!} = \frac{(k+1)_m}{m!}, \tag{5.10}$$

to write the L-function corresponding to $Q(x) = 1/(x+a)^{m+1}$ as

$$L_Q(s) = \frac{a^{-(m+1)}}{m!}\sum_{k=0}^{\infty}\frac{(-1/a)^k(k+1)_m}{(k+1)^s}. \tag{5.11}$$

Finally use the expression (5.5) to write

$$L_Q(s) = \frac{a^{-(m+1)}}{m!}\sum_{j=1}^{m}S_1(m,j)\sum_{k=0}^{\infty}\frac{(-1/a)^k}{(k+1)^{s-j}}, \tag{5.12}$$

and identify the series as a polylogarithm to produce the result. □

Corollary 5.3. Let $m \in \mathbb{N}$. Then the integral

$$R_{m,0}(a) := \int_0^1 \frac{\log\log 1/x}{(x+a)^{m+1}}\,dx \tag{5.13}$$

is given by

$$R_{m,0}(a) = -\frac{\gamma}{a^m(1+a)m} - \frac{\gamma}{a^{m+1}m!}\sum_{j=2}^{m}\frac{S_1(m,j)T_{j-2}(1/a)}{(1+1/a)^j} - \frac{1}{a^m m!}\sum_{j=1}^{m}S_1(m,j)\text{PolyLog}^{(1,0)}[1-j,-1/a],$$

where T_j is the polynomial defined in (4.12).

Proof. The result follows from Theorem 3.2 and the expression for polylogarithms given in (4.3). □

Example 5.1. The choice $a=2$ and $m=1$ gives

$$R_{1,0}(2) := \int_0^1 \frac{\log\log 1/x\,dx}{(x+2)^2} = -\frac{\gamma}{6} - \frac{1}{2}\text{PolyLog}^{(1,0)}[0,-\tfrac{1}{2}]. \tag{5.14}$$

Mathematica 6.0 is unable to compute these integral. The expansion of the polylogarithm function gives the identity

$$\int_0^1 \frac{\log\log 1/x\,dx}{(x+2)^2} = -\frac{\gamma}{6} + \sum_{n=1}^{\infty}\frac{(-1)^n\,\log n}{2^{n+1}}. \tag{5.15}$$

6. An algorithm for the evaluation of $R_{m,j}(a)$

In this section we present an analytic expression for

$$R_{m,j}(a) := \int_0^1 \frac{x^j \,\log\log 1/x}{(x+a)^{m+1}}\,dx. \tag{6.1}$$

These results provide the evaluation of integrals of the type

$$I_Q = \int_0^1 Q(x)\log\log 1/x\,dx \tag{6.2}$$

where Q has real poles. The case of non-real poles is discussed in the next section.

Theorem 6.1. The integrals $R_{m,j}(a)$ satisfy the recurrence

$$R_{m,0}(a) = \sum_{j=0}^{r} \alpha_{j,r}(a) R_{m-r+j,j}(a), \tag{6.3}$$

for any $r \leq m$. Here

$$\alpha_{j,r}(a) := (-1)^j \binom{r}{j} a^{-r}. \tag{6.4}$$

Note 6.2. The recurrence (6.3) can now be used for increasing values of the free parameter r, to obtain analytic expressions for $R_{m,j}(a)$. For instance, $r=1$ gives

$$R_{m,0}(a) = \alpha_{0,1}(a)R_{m-1,0}(a) + \alpha_{1,1}(a)R_{m,1}(a), \tag{6.5}$$

that determines $R_{m,1}(a)$ in terms of $R_{m,0}(a)$ and $R_{m-1,0}(a)$, that were previously computed.

The proof of Theorem 6.1 employs a recurrence for the functions $\alpha_{j,r}(a)$ that is established first.

Lemma 6.3. Let $k, r \in \mathbb{N}$ and $\alpha_{j,r}(a)$ as in (6.4). Then

$$\frac{\alpha_{0,r}(a)}{(x+a)^k} + \frac{\alpha_{1,r}(a)x}{(x+a)^{k+1}} + \cdots + \frac{\alpha_{r,r}(a)x^r}{(x+a)^{k+r}} = \frac{1}{(x+a)^{k+r}}. \tag{6.6}$$

Proof. Expand the identity

$$1 = \frac{(x+a)^r}{a^r}\left(1 - \frac{x}{x+a}\right)^r.$$

□

Proof of Theorem 6.1. Multiply the relation in Lemma 6.3 by $\log\log 1/x$ and integrate over $[0,1]$.

Example 6.1. We now use the method described above to check that

$$R_{1,1}(1) := \int_0^1 \frac{x\,\log\log 1/x}{(x+1)^2}\,dx = \frac{1}{2}\left(-\log^2 2 + \gamma - \log\pi + \log 2\right). \tag{6.7}$$

The recursion (6.3) gives

$$R_{1,0}(1) = \alpha_{0,1}(1)R_{0,0}(1) + \alpha_{1,1}(1)R_{1,1}(1). \tag{6.8}$$

Using the values $\alpha_{0,1}(1) = 1$ and $\alpha_{1,1}(1) = -1$ and the integrals

$$R_{0,0}(1) = -\frac{1}{2}\log^2 2 \tag{6.9}$$

given in Example 4.1 and

$$R_{1,0}(1) = \frac{1}{2}\left(-\gamma + \log \pi - \log 2\right), \tag{6.10}$$

computed in Example 4.2, we obtain the result.

Example 6.2. The computation of $R_{1,1}(2)$ can be obtained from the recurrence

$$R_{1,0}(2) = \alpha_{0,1}(2)R_{0,0}(2) + \alpha_{1,1}(2)R_{1,1}(2), \tag{6.11}$$

and the previously computed values

$$R_{1,0}(2) = -\frac{\gamma}{6} - \frac{1}{2}\text{PolyLog}^{(1,0)}[0, -\tfrac{1}{2}], \tag{6.12}$$

and

$$R_{0,0}(2) = -\gamma \log \frac{3}{2} - \text{PolyLog}^{(1,0)}[1, -\tfrac{1}{2}]. \tag{6.13}$$

It follows that

$$\begin{aligned} R_{1,1}(2) &:= \int_0^1 \frac{x \log\log 1/x}{(x+2)^2}\, dx \\ &= \frac{\gamma}{3} - \gamma \log \frac{3}{2} + \text{PolyLog}^{(1,0)}[0, -\tfrac{1}{2}] - \text{PolyLog}^{(1,0)}[1, -\tfrac{1}{2}] \\ &= \frac{\gamma}{3} - \gamma \log \frac{3}{2} - \sum_{n=2}^{\infty} \frac{(-1)^n \log n}{2^n}(1 - 1/n). \end{aligned}$$

Example 6.3. We now illustrate the recurrence (6.3) to obtain the value

$$R_{3,2}(5) = \int_0^1 \frac{x^2 \log\log 1/x}{(x+5)^4}\, dx. \tag{6.14}$$

We first let $m = 3$ in (1.33) to obtain

$$R_{3,0}(5) = \alpha_{0,2}(5)R_{1,0}(5) + \alpha_{1,2}(5)R_{2,1}(5) + \alpha_{2,2}(5)R_{3,2}(5). \tag{6.15}$$

The integrals with second index 0 are given in (5.3) by

$$R_{1,0}(5) = -\frac{\gamma}{30} - \frac{1}{5}\text{PolyLog}^{(1,0)}[0, -\tfrac{1}{5}] \tag{6.16}$$

and

$$\begin{aligned} R_{3,0}(5) = -\frac{91\gamma}{81000} &- \frac{1}{750}\text{PolyLog}^{(1,0)}[-2, -\tfrac{1}{5}] \\ &- \frac{1}{250}\text{PolyLog}^{(1,0)}[-1, -\tfrac{1}{5}] - \frac{1}{375}\text{PolyLog}^{(1,0)}[0, -\tfrac{1}{5}]. \end{aligned}$$

The next step is to put $m = 2$ in (1.32) to obtain

$$R_{2,0}(5) = \alpha_{0,1}(5)R_{1,0}(5) + \alpha_{1,1}(5)R_{2,1}(5). \tag{6.17}$$

The values

$$R_{2,0}(5) = -\frac{11\gamma}{1800} - \frac{1}{50}\text{PolyLog}^{(1,0)}[-1, -\tfrac{1}{5}] - \frac{1}{50}\text{PolyLog}^{(1,0)}[0, -\tfrac{1}{5}]$$

and $R_{1,0}(5)$ is given in (6.16). These come from Corollary 5.3. Equation (6.17) now gives

$$R_{2,1}(5) = -\frac{\gamma}{360} + \frac{1}{10}\text{PolyLog}^{(1,0)}[-1, -\tfrac{1}{5}] - \frac{1}{10}\text{PolyLog}^{(1,0)}[0, -\tfrac{1}{5}].$$

Finally we obtain

$$R_{3,2}(5) = -\frac{\gamma}{3240} - \frac{1}{30}\text{PolyLog}^{(1,0)}[-2, -\tfrac{1}{5}] + \frac{1}{10}\text{PolyLog}^{(1,0)}[-1, -\tfrac{1}{5}] - \frac{1}{15}\text{PolyLog}^{(1,0)}[0, -\tfrac{1}{5}]$$

from (6.15).

Note 6.4. The integrals $R_{m,j}(a)$ are computable by Mathematica 6.0 for $a = 1$, but not for $a \neq 1$.

7. Evaluation of integrals with non-real poles. The integrals $C_{m,j}(a,b) = D_{m,j}(r,\theta)$

We consider now the evaluation of integrals

$$C_{m,j}(a,b) := \int_0^1 \frac{x^j}{(x^2+ax+b)^{m+1}} \log\log 1/x \, dx, \tag{7.1}$$

where $a^2 - 4b < 0$, so that the quadratic factor has non-real zeros. This is written as

$$x^2 + ax + b = (x-c)(x-\bar{c}) = x^2 - 2rx\cos\theta + r^2 \tag{7.2}$$

and we write

$$D_{m,j}(r,\theta) = \int_0^1 \frac{x^j}{(x^2 - 2rx\cos\theta + r^2)^{m+1}} \log\log 1/x \, dx. \tag{7.3}$$

Naturally $C_{m,j}(a,b) = D_{m,j}(r,\theta)$, we are simply emphasizing the polar representation of the poles.

Plan of evaluation: the computation of $D_{m,j}(r,\theta)$ can be reduced to the range $m \geq 0$ and $0 \leq j \leq 2m+1$ by dividing x^j by $(x^2 - 2rx\cos\theta + 1)^{m+1}$, in case $j \geq 2m+2$. The fact is that the recurrences (1.37) and (1.38) determine all the integrals $D_{m,j}(r,\theta)$ from $D_{0,0}(r,\theta)$ and $D_{0,1}(r,\theta)$. This is illustrated with the four integrals $D_{1,j}(r,\theta) : 0 \leq j \leq 3$. Begin with (1.37) with $m = j = 1$. This gives

$$D_{1,1}(r,\theta) = -\frac{1}{2r\sin\theta}\frac{\partial}{\partial\theta}D_{0,0}(r,\theta) \tag{7.4}$$

and then (1.38) with $m = j = 1$ gives

$$D_{1,1}(r,\theta) = \frac{1}{2\cos\theta}\left(\frac{\partial}{\partial r}D_{0,0}(r,\theta) + 2rD_{1,0}(r,\theta)\right), \tag{7.5}$$

and this determines $D_{1,0}(r,\theta)$. Now use $m = 1$, $j = 2$ in (1.37) to obtain

$$D_{1,2}(r,\theta) = -\frac{1}{2r\sin\theta}\frac{\partial}{\partial\theta}D_{0,1}(r,\theta). \tag{7.6}$$

Finally, (1.38) with $m = 1$ and $j = 3$ yields

$$D_{1,3}(r,\theta) = \frac{1}{2\cos\theta}\left(\frac{\partial}{\partial r}D_{0,2}(r,\theta) + 2rD_{1,2}(r,\theta)\right), \tag{7.7}$$

Dividing x^2 by $x^2 - 2rx\cos\theta + r^2$ expresses $D_{0,2}(r,\theta)$ as a linear combination of $D_{0,0}(r,\theta)$ and $D_{1,0}(r,\theta)$. This process determines $D_{1,3}(r,\theta)$.

We compute first the integral $D_{0,0}(r,\theta)$. This task is is divided into two cases according to whether $r = 1$ or not. Theorem 7.1 gives the result for the first case and Theorem 7.2 describes the case $r \neq 1$. The evaluation of the integrals $D_{m,j}(r,\theta)$ are then obtained by using the recurrences (1.37) and (1.38).

Calculation of $D_{0,0}(1,\theta)$. This is stated in the next theorem.

Theorem 7.1. Assume $0 < \theta < 2\pi$. Then

$$\begin{aligned} D_{0,0}(1,\theta) &:= \int_0^1 \frac{\log\log 1/x}{x^2 - 2x\cos\theta + 1}\,dx \\ &= \frac{\pi}{2\sin\theta}\left[(1-\theta/\pi)\log 2\pi + \log\left(\frac{\Gamma(1-\theta/2\pi)}{\Gamma(\theta/2\pi)}\right)\right]. \end{aligned}$$

Proof. Consider the function

$$Q(x) = \frac{\sin\theta}{x^2 - 2x\cos\theta + 1} \tag{7.8}$$

with the classical expansion

$$Q(x) = \sum_{k=0}^{\infty} \sin\left((k+1)\theta\right) x^k. \tag{7.9}$$

The corresponding L-function is given by

$$L_Q(s) = \sum_{k=0}^{\infty} \frac{\sin\left((k+1)\theta\right)}{(k+1)^s}, \tag{7.10}$$

and its value at $s = 1$ is given by

$$L_Q(1) = \sum_{k=0}^{\infty} \frac{\sin\left((k+1)\theta\right)}{k+1} = \frac{\pi-\theta}{2}, \tag{7.11}$$

while the closed form of the derivative at $s = 1$ is

$$\begin{aligned} L_Q'(1) &= -\sum_{k=0}^{\infty} \frac{\sin\left((k+1)\theta\right)}{k+1}\log(k+1) \\ &= -\frac{\pi}{2}\left(\log\left(\frac{\Gamma(\theta/2\pi)}{\Gamma(1-\theta/2\pi)}\right) + (\gamma + \log 2\pi)\left(\frac{\theta}{\pi} - 1\right)\right). \end{aligned} \tag{7.12}$$

This identity can be found in [2], page 250, #30. The result now follows from Theorem 3.2. □

Calculation of $D_{0,0}(r,\theta)$ **in the case** $r \neq 1$. This is stated in the theorem below.

Theorem 7.2. Assume $0 < \theta < 2\pi$ and $r \neq 1$. Then

$$\begin{aligned} D_{0,0}(r,\theta) &:= \int_0^1 \frac{\log\log 1/x}{x^2 - 2rx\cos\theta + r^2}\,dx \\ &= -\frac{\gamma}{r\sin\theta}\tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right) \\ &\quad + \frac{1}{2ri\sin\theta}\left(\text{PolyLog}^{(1,0)}[1, e^{i\theta}/r] - \text{PolyLog}^{(1,0)}[1, e^{-i\theta}/r]\right). \end{aligned}$$

Proof. Consider the function

$$Q(x) = \frac{r^2\sin\theta}{x^2 - 2rx\cos\theta + r^2} = \sum_{k=0}^{\infty} \frac{\sin\left((k+1)\theta\right)}{r^k} x^k \tag{7.13}$$

and the corresponding L-function

$$\begin{aligned} L_Q(s) &:= \sum_{k=0}^{\infty} \frac{\sin\left((k+1)\theta\right)}{r^k (k+1)^s} \\ &= \sum_{k=0}^{\infty} \frac{e^{i(k+1)\theta} - e^{-i(k+1)\theta}}{2ir^k(k+1)^s} \\ &= \frac{r}{2i}\left(\text{PolyLog}[s, e^{it}/r] - \text{PolyLog}[s, e^{-it}/r]\right). \end{aligned}$$

The identity

$$\text{PolyLog}[1, a] = -\log(1-a), \tag{7.14}$$

yields the value

$$L_Q(1) = \frac{r}{2i}\left[-\log(1 - e^{i\theta}/r) + \log(1 - e^{-i\theta}/r)\right]. \tag{7.15}$$

The value of $L_Q(1)$ can be written as

$$L_Q(1) = r\tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right). \tag{7.16}$$

This follows by checking that both expressions for $L_Q(1)$ match at $\theta = 0$ and their derivatives with respect to θ match. □

We now present several special cases of these evaluations. Many of them appear in the table of integrals [8].

Example 7.1. The value $D_{0,0}(1,\theta)$ in Theorem 7.1 appears as 4.325.7 in [8].

Example 7.2. Replacing θ by $\theta + \pi$, we obtain the evaluation

$$\int_0^1 \frac{\log\log 1/x}{x^2 + 2x\cos\theta + 1}\,dx = \frac{\pi}{2\sin\theta}\left[\frac{\theta\log 2\pi}{\pi} + \log\left(\frac{\Gamma(1/2 + \theta/2\pi)}{\Gamma(1/2 - \theta/2\pi)}\right)\right].$$

This appears as 4.231.2 in [8].

Example 7.3. The value $\theta = \pi/2$ provides

$$D_{0,0}\left(1, \frac{\pi}{2}\right) = \int_0^1 \frac{\log\log 1/x}{1+x^2}\,dx = \frac{\pi}{4}\log 2\pi + \frac{\pi}{2}\log\frac{\Gamma(3/4)}{\Gamma(1/4)}.$$

This is the example discussed by Vardi in [13].

Example 7.4. The angle $\theta = \pi/3$ yields the value

$$D_{0,0}\left(1, \frac{\pi}{3}\right) = \int_0^1 \frac{\log\log 1/x}{1-x+x^2}\,dx = \frac{2\pi\log 2\pi}{3\sqrt{3}} + \frac{\pi}{\sqrt{3}}\log\frac{\Gamma(5/6)}{\Gamma(1/6)}.$$

This appears as 4.325.6 in [8] where the answer is written in the equivalent form

$$\int_0^1 \frac{\log\log 1/x}{1-x+x^2}\,dx = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\log 2\pi - \log\Gamma(1/6)\right].$$

The equivalent form

$$\int_0^\infty \frac{\log x\,dx}{e^x+e^{-x}-1} = \frac{2\pi}{\sqrt{3}}\left[\frac{5}{6}\log 2\pi - \log\Gamma(1/6)\right]$$

appears as 4.332.1 in [8].

Example 7.5. The angle $\theta = 2\pi/3$ provides an evaluation of 4.325.5 in [8]:

$$D_{0,0}\left(1, \frac{2\pi}{3}\right) = \int_0^1 \frac{\log\log 1/x}{1+x+x^2}\,dx = \frac{\pi\log 2\pi}{3\sqrt{3}} + \frac{\pi}{\sqrt{3}}\log\frac{\Gamma(2/3)}{\Gamma(1/3)}.$$

The equivalent form

$$\int_0^\infty \frac{\log x\,dx}{e^x+e^{-x}+1} = \frac{\pi}{\sqrt{3}}\log\left(\frac{\Gamma(2/3)\sqrt{2\pi}}{\Gamma(1/3)}\right) \tag{7.17}$$

appears *incorrectly* as 4.332.2 in [8]. The correct result is obtained by replacing $\sqrt{2\pi}$ by $\sqrt[3]{2\pi}$.

Example 7.6. The limit of $D_{0,0}(1,\theta)$ as $\theta \to \pi$ gives the evaluation of

$$D_{0,0}\left(1, \frac{\pi}{2}\right) = \int_0^1 \frac{\log\log 1/x}{(1+x)^2}\,dx = \frac{1}{2}\left(\log\pi - \log 2 - \gamma\right).$$

This is 4.325.3 of [8].

Example 7.7. It is easy to choose an angle and produce an integral that cannot be evaluated by Mathematica 6.0. For example, $\theta = 3\pi/4$ gives

$$D_{0,0}\left(1, \frac{3\pi}{4}\right) = \int_0^1 \frac{\log\log 1/x}{1+\sqrt{2}\,x+x^2}\,dx = \frac{\pi}{\sqrt{2}}\left(\frac{\log 2\pi}{4} + \log\left(\frac{\Gamma(5/8)}{\Gamma(3/8)}\right)\right).$$

Note 7.3. The evaluation of $L_Q(1)$ yields the identity

$$\int_0^1 \frac{dx}{x^2 - 2rx\cos\theta + r^2} = \frac{1}{r\sin\theta}\tan^{-1}\left(\frac{\sin\theta}{r-\cos\theta}\right). \tag{7.18}$$

Note 7.4. The polylogarithm terms appearing in the expression for $D_{0,0}(r,\theta)$ can be written as in terms of the sum

$$U(r,\theta) := \sum_{k=0}^{\infty} \sin\left((k+1)\theta\right)\,\frac{\log(k+1)}{r^k(k+1)}. \tag{7.19}$$

The authors are unable to express the function $U(r,\theta)$ in terms of special functions with real arguments.

We now proceed to a systematic determination of the integrals $D_{m,j}(r,\theta)$ for $m,\, j > 0$. For that, we use the recurrences (1.37) and (1.38).

Proposition 7.5. Assume $0 < \theta < 2\pi$. Then

$$\begin{aligned} D_{1,1}(1,\theta) &:= \int_0^1 \frac{x \, \log\log 1/x}{(x^2-2x\cos\theta+1)^2}\,dx \\ &= \frac{\log 2\pi}{4\sin^2\theta}(1+(\pi-\theta)\cot\theta) + \frac{1}{8\sin^2\theta}\left(\psi\left(\theta/2\pi\right)+\psi\left(1-\theta/2\pi\right)\right) \\ &\quad + \frac{\pi}{4}\csc^2\theta\cot\theta\log\left(\frac{\Gamma(1-\theta/2\pi)}{\Gamma(\theta/2\pi)}\right). \end{aligned}$$

Proof. The result follows directly from (1.37) and Theorem 7.1. □

Note 7.6. For the case $r \neq 1$ the value of $D_{1,1}(r,\theta)$ can be obtained by differentiating the expression for $D_{0,0}(r,\theta)$ in Theorem 7.2.

Particular cases of this result are stated next.

Example 7.8. The angle $\theta = \pi/2$ produces

$$D_{1,1}\left(1,\frac{\pi}{2}\right) := \int_0^1 \frac{x \, \log\log 1/x}{(x^2+1)^2}\,dx = \frac{\log 2\pi}{4} + \frac{1}{8}\psi\left(\frac{1}{4}\right) + \frac{1}{8}\psi\left(\frac{3}{4}\right).$$

Here $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the polygamma function. Using the values

$$\psi\left(\frac{1}{4}\right) = -\gamma - \frac{\pi}{2} - 3\log 2 \text{ and } \psi\left(\frac{3}{4}\right) = -\gamma + \frac{\pi}{2} - 3\log 2$$

that appear in [8] as 8.366.4 and 8.366.5 respectively, we obtain

$$\int_0^1 \frac{x \, \log\log 1/x}{(x^2+1)^2}\,dx = \frac{1}{4}(\log\pi - 2\log 2 - \gamma). \tag{7.20}$$

The equivalent version

$$\int_0^\infty \frac{\log x \, dx}{\cosh^2 x} = \log\pi - 2\log 2 - \gamma, \tag{7.21}$$

appears as 4.371.3 in [8].

Example 7.9. The angle $\theta = \pi/3$ gives the evaluation

$$\begin{aligned} D_{1,1}\left(1,\frac{\pi}{3}\right) &:= \int_0^1 \frac{x \, \log\log 1/x}{(x^2-x+1)^2}\,dx \\ &= \frac{\log 2\pi}{3} + \frac{2\pi\log 2\pi}{9\sqrt{3}} + \frac{\pi}{3\sqrt{3}}\log\left(\frac{\Gamma(5/6)}{\Gamma(1/6)}\right) \\ &\quad + \frac{1}{6}\psi\left(\frac{1}{6}\right) + \frac{1}{6}\psi\left(\frac{5}{6}\right). \end{aligned}$$

Using elementary properties of the ψ function and the values

$$\psi\left(\frac{1}{6}\right) = -\gamma - \frac{\pi\sqrt{3}}{2} - \frac{3}{2}\log 3 - 2\log 2, \tag{7.22}$$

and

$$\psi\left(\frac{5}{6}\right) = -\gamma + \frac{\pi\sqrt{3}}{2} - \frac{3}{2}\log 3 - 2\log 2, \tag{7.23}$$

that appear in [12], page 21, we obtain

$$\int_0^1 \frac{x \log\log 1/x}{(x^2-x+1)^2}\,dx = -\frac{\gamma}{3} - \frac{\log 2}{3} + \frac{7\pi\log 2}{9\sqrt{3}} - \frac{\log 3}{2} - \frac{\pi \log 3}{3\sqrt{3}} + \frac{\log\pi}{3} + \frac{8\pi\log\pi}{9\sqrt{3}} - \frac{4\pi}{3\sqrt{3}}\log\Gamma\left(\frac{1}{3}\right).$$

Example 7.10. The recurrences (1.37) and (1.38) yield

$$\int_0^1 \frac{x^2 \log\log 1/x}{(x^2-\sqrt{3}x+1)^3}\,dx = \frac{1}{6}(9\sqrt{3}+25\pi)\log 2\pi + 5\pi\log\left(\frac{\Gamma(11/12)}{\Gamma(1/12)}\right) + \frac{3\sqrt{3}}{4}\left(\psi(1/12)+\psi(11/12)\right) + \frac{1}{8\pi}\left(\psi'(11/12)-\psi'(1/12)\right).$$

This can be written as

$$\int_0^1 \frac{x^2 \log\log 1/x}{(x^2-\sqrt{3}x+1)^3}\,dx = -\frac{3\sqrt{3}\gamma}{2} - 3\sqrt{3}\log 2 + \frac{35}{3}\pi\log 2 - \frac{9}{4}\sqrt{3}\log 3 + \frac{9}{2}\log(2-\sqrt{3}) - 5\pi\log(\sqrt{3}-1) + \frac{3\sqrt{3}}{2}\log\pi + \frac{55}{6}\pi\log\pi - 10\pi\log\Gamma(1/12) - \frac{1}{8\pi}\psi'(1/12) + \frac{1}{8\pi}\psi'(11/12).$$

Example 7.11. The values $r=2$ and $\theta=\pi/3$ yields the evaluation

$$\int_0^1 \frac{\log\log 1/x\,dx}{x^2-2x+4} = -\frac{\gamma\pi}{6\sqrt{3}} - \frac{i}{2\sqrt{3}}\left(\text{PolyLog}^{(1,0)}\left[1,\tfrac{1+i\sqrt{3}}{4}\right] - \text{PolyLog}^{(1,0)}\left[1,\tfrac{1-i\sqrt{3}}{4}\right]\right).$$

Mathematica 6.0 is unable to evaluate this integral.

Calculation of $D_{0,1}(r,\theta)$.

The integral

$$D_{0,1}(r,\theta) = \int_0^1 \frac{x\log\log 1/x\,dx}{x^2-2rx\cos\theta+r^2} \tag{7.24}$$

corresponds to

$$Q(x) = \frac{x}{x^2-2rx\cos\theta+r^2}. \tag{7.25}$$

To evaluate the integral $D_{0,1}(r,\theta)$ we employ the expansion

$$\frac{x}{x^2-2rx\cos\theta+r^2} = \sum_{k=0}^{\infty} \frac{\sin k\theta}{r^{k+1}\sin\theta}x^k. \tag{7.26}$$

We conclude that the L-function associated to this Q is

$$L_Q(s) = \frac{1}{\sin\theta}\sum_{k=0}^{\infty} \frac{\sin k\theta}{r^{k+1}(k+1)^s}. \tag{7.27}$$

Therefore

$$L_Q(1) = \frac{1}{\sin\theta}\sum_{k=0}^{\infty} \frac{\sin(k+1)\theta}{r^{k+1}(k+2)}. \tag{7.28}$$

To evaluate this sum, integrate (7.26) from 0 to 1 to produce

$$L_Q(1) = \int_0^1 \frac{x\,dx}{x^2 - 2rx\cos\theta + r^2}. \tag{7.29}$$

Observe that

$$\begin{aligned} \int_0^1 \frac{x\,dx}{x^2 - 2rx\cos\theta + r^2} &= \frac{1}{2}\int_0^1 \frac{(2x - 2r\cos\theta)\,dx}{x^2 - 2rx\cos\theta + r^2} + \\ &+ r\cos\theta \int_0^1 \frac{dx}{x^2 - 2rx\cos\theta + r^2}. \end{aligned}$$

Both integrals are elementary, the latter is given in (7.18). Therefore,

$$L_Q(1) = \frac{1}{2}\log\left(\frac{r^2 - 2r\cos\theta + 1}{r^2}\right) + \cot\theta\,\tan^{-1}\left(\frac{\sin\theta}{r - \cos\theta}\right). \tag{7.30}$$

The L-series (7.27) can be expressed in terms of the Lerch Φ-function

$$\Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}. \tag{7.31}$$

Indeed,

$$\begin{aligned} L_Q(s) &= \frac{1}{2i\sin\theta}\sum_{k=0}^{\infty} \frac{e^{ik\theta} - e^{-ik\theta}}{r^{k+1}\,(k+1)^s} \\ &= \frac{1}{2ir\sin\theta}\left(\sum_{k=0}^{\infty} \frac{(e^{i\theta}/r)^k}{(k+1)^s} - \frac{(e^{-i\theta}/r)^k}{(k+1)^s}\right) \\ &= \frac{1}{2ri\sin\theta}\left[\Phi\left(\frac{e^{i\theta}}{r}, s, 1\right) - \Phi\left(\frac{e^{-i\theta}}{r}, s, 1\right)\right]. \end{aligned}$$

The next statement gives the value of $D_{0,1}(r,\theta)$.

Theorem 7.7. The integral

$$D_{0,1}(r,\theta) = \int_0^1 \frac{x\log\log 1/x\,dx}{x^2 - 2xr\cos\theta + r^2} \tag{7.32}$$

is given by

$$\begin{aligned} D_{0,1}(r,\theta) &= -\frac{\gamma}{2}\log\left(\frac{r^2 - 2r\cos\theta + 1}{r^2}\right) \\ &- \gamma\cot\theta\,\tan^{-1}\left(\frac{\sin\theta}{r - \cos\theta}\right) \\ &+ \frac{1}{2ri\sin\theta}\left[\Phi^{(0,1,0)}\left(\frac{e^{i\theta}}{r}, 1, 1\right) - \Phi^{(0,1,0)}\left(\frac{e^{-i\theta}}{r}, 1, 1\right)\right]. \end{aligned}$$

Note 7.8. This evaluation completes the algorithm to evaluate all the integrals $D_{m,j}(r,\theta)$.

8. Adamchik's integrals

V. Adamchik presented in [1] a series of beautiful evaluations of integrals of the form (1.1), where the denominator has the form $(1+x^n)^m$ for $n, m \in \mathbb{N}$. The results are expressed in terms of the Hurwitz zeta function and its derivatives. For example, in Proposition 3, it is shown that

$$\begin{aligned}\int_0^1 \frac{x^{p-1}}{1+x^n} \log\log 1/x \, dx &= \frac{\gamma+\log 2n}{2n}\left(\psi\left(\frac{p}{2n}\right)-\psi\left(\frac{n+p}{2n}\right)\right) \\ &+ \frac{1}{2n}\left(\zeta'\left(1,\frac{p}{2n}\right)-\zeta'\left(1,\frac{n+p}{2n}\right)\right),\end{aligned}$$

followed by Proposition 4 that states that

$$\begin{aligned}\int_0^1 x^{p-1}\frac{1-x}{1-x^n} \log\log 1/x \, dx &= \frac{\gamma+\log n}{n}\left(\psi\left(\frac{p}{n}\right)-\psi\left(\frac{p+1}{n}\right)\right) \\ &+ \frac{1}{n}\left(\zeta'\left(1,\frac{p}{n}\right)-\zeta'\left(1,\frac{p+1}{n}\right)\right).\end{aligned}$$

The expressions become more complicated as the exponent of the denominator increases. For instance, Proposition 5 gives

$$\begin{aligned}\int_0^1 \frac{x^{p-1}}{(1+x^n)^2} \log\log 1/x \, dx &= \frac{(n-p)}{2n^2}(\log 2n+\gamma)\left(\psi\left(\frac{p}{2n}\right)-\psi\left(\frac{n+p}{2n}\right)\right) \\ &- \frac{1}{2n}\left(\gamma+\log 2n-2\log\left(\frac{\Gamma(\frac{p}{2n})}{\Gamma(\frac{n+p}{2n})}\right)\right) \\ &+ \frac{n-p}{2n^2}\left(\zeta'\left(1,\frac{p}{2n}\right)-\zeta'\left(1,\frac{n+p}{2n}\right)\right),\end{aligned}$$

and in Proposition 6 we find

$$\begin{aligned}\int_0^1 \frac{x^{p-1}}{(1+x^n)^3} \log\log 1/x \, dx &= \frac{3n-2p}{2n^2}\log\left(\frac{\Gamma\left(\frac{p}{2n}\right)}{\Gamma\left(\frac{n+p}{2n}\right)}\right) - \frac{(5n-2p)(\log 2n+\gamma)}{8n^2} \\ &+ \frac{(n-p)(2n-p)(\log 2n+\gamma)}{4n^2}\left(\psi\left(\frac{p}{2n}\right)-\psi\left(\frac{n+p}{2n}\right)\right) \\ &+ \frac{1}{n}\left(\zeta'\left(-1,\frac{p}{2n}\right)-\zeta'\left(-1,\frac{n+p}{2n}\right)\right) \\ &+ \frac{(n-p)(2n-p)}{4n^3}\left(\zeta'\left(1,\frac{p}{2n}\right)-\zeta'\left(1,\frac{n+p}{2n}\right)\right).\end{aligned}$$

We now describe some examples on how to use Theorem 3.2 to obtain some of the specific examples in [1].

Example 8.1. From Proposition 3 it follows that

$$\int_0^1 \frac{x^{n-1}}{1+x^n} \log\log 1/x \, dx = -\frac{\log 2\,\log 2n^2}{2n}. \tag{8.1}$$

This appears as (27) in [1]. To check this evaluation, observe that

$$Q(x) = \frac{x^{n-1}}{1+x^n} = \sum_{k=1}^{\infty} (-1)^{k-1} x^{kn-1}, \tag{8.2}$$

so the corresponding L-function is

$$L_Q(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s n^s} = (1-2^{-s}) \frac{\zeta(s)}{n^s}. \tag{8.3}$$

A direct calculation shows that $L_Q(1) = \frac{\log 2}{n}$ and

$$L_Q'(1) = \frac{\gamma \log 2}{n} - \frac{\log^2 2}{2n} - \frac{\log 2 \, \log n}{n}. \tag{8.4}$$

Then (8.1) follows from (3.5).

Example 8.2. Formula (28) in [1] is also obtained from Proposition 3 and it states that

$$\int_0^1 \frac{x^{2n-1}}{1+x^n} \log\log 1/x \, dx = \frac{1}{2n} \left(\log^2 2 + 2(\log 2 - 1) \log n - 2\gamma \right). \tag{8.5}$$

To prove this, consider

$$Q(x) = \frac{x^{2n-1}}{1+x^n} = \sum_{k=0}^{\infty} (-1)^k x^{(2+k)n-1}, \tag{8.6}$$

whose L-function is

$$L_Q(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2+k)^s n^s} = \frac{1}{n^s} \left(1 - (1-2^{1-s})\zeta(s) \right). \tag{8.7}$$

Replacing the values

$$L_Q(1) = \frac{1-\log 2}{n} \text{ and } L_Q'(1) = \frac{1}{2n} \left(\log^2 2 - 2\gamma \log 2 - 2 \log n + 2 \log 2 \log n \right),$$

in (3.5) we obtain the result.

Example 8.3. The identity

$$\int_0^1 \frac{x}{1+x^4} \log\log 1/x \, dx = \frac{\pi}{4} \log \left(\frac{\sqrt{\pi}\, \Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right), \tag{8.8}$$

appears as formula (30) in [1]. To establish it, consider the function

$$Q(x) = \frac{x}{1+x^4} = \sum_{k=0}^{\infty} (-1)^k x^{4k+1}, \tag{8.9}$$

with L-function

$$L_Q(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+2)^s} = \frac{1}{2^{3s}} \left(\zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{3}{4}) \right). \tag{8.10}$$

The result follows from Theorem 3.2 by using the values

$$L_Q(1) = \tfrac{1}{8} \left(\psi\left(\tfrac{3}{4}\right) - \psi\left(\tfrac{1}{4}\right) \right), \tag{8.11}$$

and

$$L_Q'(1) = -\tfrac{3 \log 2}{8} \left(\psi\left(\tfrac{3}{4}\right) - \psi\left(\tfrac{1}{4}\right) \right) + \tfrac{1}{8} \left(\zeta'(1, \tfrac{1}{4}) - \zeta'(1, \tfrac{3}{4}) \right). \tag{8.12}$$

The special values

$$\psi(\tfrac{1}{4}) = -\gamma - \tfrac{\pi}{2} - 3 \log 2 \text{ and } \psi(\tfrac{3}{4}) = -\gamma + \tfrac{\pi}{2} - 3 \log 2, \tag{8.13}$$

and the value

$$\zeta'(1,\tfrac{1}{4}) - \zeta'(1,\tfrac{3}{4}) = \pi\left(\gamma + \log 2 + 3\log 2\pi - 4\log\Gamma\left(\tfrac{1}{4}\right)\right), \tag{8.14}$$

are used to simplified the result. This last expression comes from

$$\begin{aligned}\zeta'(1,\tfrac{p}{q}) - \zeta'(1,1-\tfrac{p}{q}) &= \pi\cot\frac{\pi p}{q}\left(\log 2\pi q + \gamma\right) \\ &- 2\pi\sum_{j=1}^{q-1}\log\left(\Gamma\left(\tfrac{j}{q}\right)\right)\sin\frac{2\pi jp}{q}.\end{aligned}$$

This identity, established in [1], follows directly from the classical Rademacher formula

$$\zeta\left(z,\tfrac{p}{q}\right) = 2\Gamma(1-z)(2\pi q)^{z-1}\sum_{j=1}^{q}\sin\left(\tfrac{\pi z}{2} + \tfrac{2jpz}{q}\right)\zeta\left(1-z,\tfrac{j}{q}\right). \tag{8.15}$$

An alternative evaluation of this integral comes from the partial fraction decomposition

$$\frac{x}{x^4+(2-a^2)x^2+1} = \frac{1}{2a}\frac{1}{x^2-ax+1} - \frac{1}{2a}\frac{1}{x^2+ax+1}. \tag{8.16}$$

We assume $|a|<2$ and write $a = 2\cos\theta$. Then (8.16) yields

$$\int_0^1 \frac{x\,\log\log 1/x}{x^4+(2-4\cos^2\theta)x^2+1}\,dx = \frac{1}{4\cos\theta}\left(D_{0,0}(1,\theta) - D_{0,0}(1,\pi+\theta)\right). \tag{8.17}$$

Using the result of Theorem 7.1 we obtain

$$\begin{aligned}\int_0^1 \frac{x\,\log\log 1/x\,dx}{x^4+(2-4\cos^2\theta)x^2+1} &= \frac{\pi}{4\sin 2\theta}\times \\ \left(\log\left(\frac{4\pi^3}{\sin\theta}\right) - \frac{2\theta}{\pi}\log 2\pi\right. &\left. - 2\log\left[\Gamma\left(\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi}\right)\right]\right).\end{aligned} \tag{8.18}$$

The special case $\theta = \pi/4$ produces (8.8).

Example 8.4. The case $\theta = \pi/2$ in the previous example reduces to Example 7.8.

Example 8.5. The angle $\theta = \pi/3$ in (8.18) yields

$$\int_0^1 \frac{x\,\log\log 1/x}{x^4+x^2+1}\,dx = \frac{\pi}{12\sqrt{3}}\left(6\log 2 - 3\log 3 + 8\log\pi - 12\log\Gamma\left(\tfrac{1}{3}\right)\right). \tag{8.19}$$

Example 8.6. The angle $\theta = \pi/8$ in (8.18) yields

$$\int_0^1 \frac{x\,\log\log 1/x}{x^4-\sqrt{2}x^2+1}\,dx = \frac{\pi}{8\sqrt{2}}\left(7\log\pi - 4\log\sin\tfrac{\pi}{8} - 8\log\Gamma\left(\tfrac{1}{8}\right)\right). \tag{8.20}$$

9. A hyperbolic example

The method introduced here can be used to provide an analytic expression for the family

$$LC_n := \int_0^\infty \frac{\log t\,dt}{\cosh^{n+1} t}. \tag{9.1}$$

The table of integrals [8] contains

$$LC_0 = \int_0^\infty \frac{\log t\, dt}{\cosh t} = \frac{\pi}{2}\left(2\log 2 + 3\log \pi - 4\log\Gamma\left(\tfrac{1}{4}\right)\right), \tag{9.2}$$

as formula 4.371.1 and

$$LC_1 = \int_0^\infty \frac{\log t\, dt}{\cosh^2 t} = -\gamma + \log\pi - 2\log 2, \tag{9.3}$$

as 4.371.3.

The change of variables $x = e^{-t}$ shows that

$$LC_n = 2^{n+1}\int_0^1 \frac{x^n \, \log\log 1/x}{(x^2+1)^{n+1}}\, dx, \tag{9.4}$$

that identifies this integral as

$$LC_n = 2^{n+1} D_{n,n}\left(1, \tfrac{\pi}{2}\right) \tag{9.5}$$

The recurrence (1.37), for $r = 1$ and $j = m$, become

$$D_{m,m}(1,\theta) = -\frac{1}{2m\sin\theta}\frac{\partial}{\partial\theta} D_{m-1,m-1}(1,\theta), \tag{9.6}$$

and the initial condition

$$D_{0,0}(1,\theta) = \frac{\pi}{2\sin\theta}\left[(1-\theta/\pi)\log 2\pi + \log\left(\frac{\Gamma(1-\theta/2\pi)}{\Gamma(\theta/2\pi)}\right)\right],$$

provides a systematic procedure to compute LC_n. For instance, it follows that

$$\begin{aligned} LC_2 &= 2^3 D_{2,2}(1, \tfrac{\pi}{2}) \\ &= -\frac{2}{\pi}\text{Catalan} + \frac{\pi}{4}\left(2\log 2 + 3\log\pi - 4\log\Gamma\left(\tfrac{1}{4}\right)\right), \end{aligned}$$

and

$$\begin{aligned} LC_3 &= 2^4 D_{3,3}(1, \tfrac{\pi}{2}) \\ &= -\frac{2\gamma}{3} - \frac{4\log 2}{3} + \frac{2\log\pi}{3} + \frac{28}{3}\zeta'(-2). \end{aligned}$$

The *Catalan constant* appearing above is defined by

$$\text{Catalan} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}. \tag{9.7}$$

10. Small sample of a new type of evaluations

We have introduced here a systematic method to deal with integrals of the form

$$I_Q = \int_0^1 Q(x)\log\log 1/x\, dx. \tag{10.1}$$

Extensions of this technique provides examples such as

$$\begin{aligned}
\int_0^1 (1+\log x)\log(x+1)\,\log\log 1/x\,dx &= 1+(\gamma-1)\left(\frac{\pi^2}{12}-1\right) \\
&\quad -\left(\frac{\pi^2}{12}+2\right)\log 2-\frac{\zeta'(2)}{2}, \\
\int_0^1 (1+\log x)\,\tan^{-1} x\,\log\log 1/x\,dx &= (1-\gamma)\frac{\pi^2}{48}-\frac{\pi}{4}+\frac{\log 2}{2}+\frac{\zeta'(2)}{8}, \\
\int_0^1 \frac{\tanh^{-1}\sqrt{x}}{x}\,\log\log 1/x\,dx &= -\frac{\gamma\pi^2}{4}+\frac{\pi^2\log 2}{3}+\frac{3}{2}\zeta'(2).
\end{aligned}$$

Details will presented elsewhere.

11. Conclusions

We have developed an algorithm to evaluate integrals of the form

$$I_Q = \int_0^1 Q(x)\,\log\log 1/x\,dx. \tag{11.1}$$

In the case where $Q(x)$ is analytic at $x=0$, with power series expansion

$$Q(x) = \sum_{n=0}^{\infty} a_n x^n \tag{11.2}$$

we associate to Q its L-function

$$L_Q(s) = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^s}. \tag{11.3}$$

Then

$$I_Q = -\gamma L_Q(1) + L_Q'(1). \tag{11.4}$$

In the case $Q(x)$ is a rational function, we provide explicit expressions for I_Q in terms of special values of the logarithm, the Riemann zeta function, the polylogarithm PolyLog$[c,x]$, its first derivative with respect to c and the Lerch Φ-function.

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