The integrals in Gradshteyn and Ryzhik.
Part 23: Combination of logarithms and rational functions

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Abstract. The table of Gradshteyn and Ryzhik contains many entries where the integrand is a combination of a rational function and a logarithmic function. The proofs presented here, complete the evaluation of all entries in Section 4.231 and 4.291.

1. Introduction

The table of integrals [6] contains many entries of the form

\[(1.1) \quad \int_{a}^{b} R_1(x) \ln R_2(x) \, dx\]

where \(R_1\) and \(R_2\) are rational functions. Some of these examples have appeared in previous papers: entry 4.291.1

\[(1.2) \quad \int_{0}^{1} \frac{\ln(1 + x)}{x} \, dx = \frac{\pi^2}{12}\]

as well as entry 4.291.2

\[(1.3) \quad \int_{0}^{1} \frac{\ln(1 - x)}{x} \, dx = -\frac{\pi^2}{6}\]

have been established in [4], entry 4.212.7

\[(1.4) \quad \int_{1}^{e} \frac{\ln x \, dx}{(1 + \ln x)^2} = \frac{e}{2} - 1\]

appears in [2] and entry 4.231.11

\[(1.5) \quad \int_{0}^{a} \frac{\ln x \, dx}{x^2 + a^2} = \frac{\pi \ln a}{4a} - \frac{G}{a}\]

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where
\[(1.6)\quad G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\]
is the Catalan constant, has appeared in [5]. The value of entry 4.233.1
\[(1.7)\quad \int_0^1 \frac{\ln x}{x^2 + x + 1} \, dx = \frac{2}{3} \left[ \frac{2\pi^2}{3} - \psi\left(\frac{1}{3}\right) \right],\]
where \(\psi(x) = \Gamma'(x)/\Gamma(x)\) is the digamma function, was established in [8].

A standard trick employed in the evaluations of integrals over \([0, \infty)\), is to transform the interval \([1, \infty)\) back to \([0, 1]\) via \(t = 1/x\). This gives
\[(1.8)\quad \int_0^\infty R(x) \ln x \, dx = \int_0^1 \left[ R(x) - \frac{1}{x^2} R\left(\frac{1}{x}\right) \right] \, dx.\]
In particular, if the rational function satisfies
\[(1.9)\quad R\left(\frac{1}{x}\right) = x^2 R(x),\]
then
\[(1.10)\quad \int_0^\infty R(x) \ln x \, dx = 0.\]
This is the case for \(R(x) = \frac{1 + x^2}{(1 - x^2)^2}\) and (1.10) appears as entry 4.234.3 in [6].

The goal of this paper is to present a systematic evaluation of the entries in [6] of the form (1.1).

2. Combinations of logarithms and linear rational functions

**Example 2.1.** Entry 4.291.3 states that
\[(2.1)\quad \int_0^{1/2} \frac{\ln(1-x)}{x} \, dx = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}.\]
To evaluate this integral let \(t = -\ln(1-x)\) to produce
\[(2.2)\quad \int_0^{1/2} \frac{\ln(1-x)}{x} \, dx = - \int_0^{\ln 2} \frac{te^{-t}}{1-e^{-t}} \, dt.\]
This last integral can be written as
\[(2.3)\quad \int_0^{\ln 2} t \, dt - \int_0^{\ln 2} \frac{t \, dt}{1-e^{-t}}.\]
The first integral is elementary and has value \(\frac{1}{2}\ln^2 2\). The second integral was evaluated as \(\pi^2/12\) in [3].
Example 2.2. The change of variables $t = x/2$ converts (2.1) to
\begin{equation}
\int_0^{1/2} \ln \left(1 - \frac{t}{2}\right) \frac{dt}{t} = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}.
\end{equation}
This is entry 4.291.4 of [6].

Example 2.3. Entry 4.291.5 states that
\begin{equation}
\int_0^1 \ln \left(\frac{1 + x}{2}\right) \frac{dx}{1 - x} = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}.
\end{equation}
To evaluate this entry, let $u = (1 - x)/2$ to reduce it to (2.1)

Example 2.4. Differentiating
\begin{equation}
\int_0^1 \ln(1 + x) \frac{dx}{1 + x} = \frac{2^{-a}(2^a - 2)}{a - 1}
\end{equation}
with respect to $a$ gives
\begin{equation}
\int_0^1 (1 + x)^{-a} \ln(1 + x) dx = \frac{1}{(a - 1)^2} \left(2^{-a}(-2 + 2^a + 2 \ln 2 - 2a \ln 2)\right).
\end{equation}
Now let $a \to 1$ to obtain
\begin{equation}
\int_0^1 \frac{\ln(1 + x)}{1 + x} dx = \frac{1}{2} \ln^2 2.
\end{equation}
This is entry 4.291.6.

Example 2.5. The partial fraction decomposition
\begin{equation}
\frac{1}{x(1 + x)} = \frac{1}{x} - \frac{1}{1 + x}
\end{equation}
gives
\begin{equation}
\int_0^1 \frac{\ln(1 + x)}{x(1 + x)} dx = \int_0^1 \frac{\ln(1 + x)}{x} dx - \int_0^1 \frac{\ln(1 + x)}{1 + x}.
\end{equation}
The first integral is entry 4.291.1 and it has value $\pi^2/12$ as shown in [4]. The second integral is $\frac{1}{2} \ln^2 2$ as established in Example 2.4. This gives entry 4.291.12
\begin{equation}
\int_0^1 \frac{\ln(1 + x)}{x(1 + x)} dx = \frac{\pi^2}{12} - \frac{1}{2} \ln^2 2.
\end{equation}

Example 2.6. Entry 4.291.13 is
\begin{equation}
\int_0^\infty \frac{\ln(1 + x)}{x(1 + x)} dx = \frac{\pi^2}{6}.
\end{equation}
Split the integral over $[0, 1]$ and $[1, \infty)$ and make the change of variables $t = 1/x$ in the second part. This gives
\begin{equation}
\int_0^\infty \frac{\ln(1 + x)}{x(1 + x)} dx = \int_0^1 \frac{\ln(1 + x)}{x(1 + x)} dx + \int_0^1 \frac{\ln(1 + t) - \ln t}{1 + t} dt.
\end{equation}
Expand the first integral in partial fractions to obtain
\[ (2.14) \quad \int_0^\infty \frac{\ln(1 + x)}{x(1 + x)} \, dx = \int_0^1 \frac{\ln(1 + x)}{x} \, dx - \int_0^1 \frac{\ln x}{1 + x} \, dx. \]
Integrate by parts the second integral to obtain
\[ (2.15) \quad \int_0^\infty \frac{\ln(1 + x)}{x(1 + x)} \, dx = 2 \int_0^1 \frac{\ln(1 + x)}{x} \, dx. \]
The evaluation
\[ (2.16) \quad \int_0^1 \frac{\ln(1 + x)}{x} \, dx = \frac{\pi^2}{12} \]
that appears as entry 4.291.1 has been established in [4].

3. Combinations of logarithms and rational functions with denominators that are squares of linear terms

This section evaluates integrals of the form
\[ (3.1) \quad \int_a^b R_2(x) \ln R_1(x) \, dx \]
where \( R_1, R_2 \) are rational functions and the denominator of \( R_2 \) is a quadratic polynomial of the form \((cx + d)^2\).

**Example 3.1.** Entry 4.291.14 is
\[ (3.2) \quad \int_0^1 \frac{\ln(1 + x)}{(ax + b)^2} \, dx = \frac{1}{a(a - b)} \ln \frac{a + b}{b} + \frac{2 \ln 2}{b^2 - a^2}, \]
and
\[ (3.3) \quad \int_0^1 \frac{\ln(1 + x)}{(x + 1)^2} \, dx = \frac{1 - \ln 2}{2} \]
gives the value when \( a = b \), after scaling.

To evaluate the first case, integrate by parts to get
\[ (3.4) \quad \int_0^1 \frac{\ln(1 + x)}{(ax + b)^2} \, dx = -\ln 2 a(a + b) + \int_0^1 \frac{dx}{a(a + x)(ax + b)}. \]
The result now follows by expanding the second integrand in partial fractions.

The case \( a = b \) is obtained by a direct integration by parts:
\[ (3.5) \quad \int_0^1 \frac{\ln(1 + x)}{(1 + x)^2} \, dx = -\frac{\ln 2}{2} + \int_0^1 \frac{dx}{(1 + x)^2}. \]
This last integral is \( 1/2 \) and the result has been established.

The same procedure gives entry 4.291.20:
\[ (3.6) \quad \int_0^1 \frac{\ln(ax + b)}{(1 + x)^2} \, dx = \frac{1}{2(a - b)} [(a + b) \ln(a + b) - 2b \ln b - 2a \ln 2], \]
for \( a \neq b \).
Example 3.2. The partial fraction decomposition
\[ \frac{1 - x^2}{(ax + b)^2 (bx + a)^2} = \frac{1}{a^2 - b^2} \left( \frac{1}{(ax + b)^2} - \frac{1}{(bx + a)^2} \right) \]
and Example 3.1 gives the evaluation of entry 4.291.25:
\[ \int_0^1 \frac{(1 - x^2) \ln(1 + x)}{(ax + b)^2 (bx + a)^2} \, dx = \frac{1}{(a^2 - b^2)(a - b)} \left[ \frac{a + b}{ab} \ln(a + b) - \frac{\ln b - \ln a}{a - b} \right] - \frac{4 \ln 2}{(a^2 - b^2)^2}. \]
The answer may be written in the more compact form
\[ \frac{-a^2 \ln a - b \ln b + a \ln(16ab)}{ab(a^2 - b^2)^2} + (a + b) \ln(a + b) \]
but this form hides the symmetry of the integral.

Example 3.3. Entry 4.291.15 is
\[ \int_0^\infty \frac{\ln(1 + x)}{(ax + b)^2} \, dx = \ln a - \ln b \]
for \( a \neq b \). In the case \( a = b \), the integral scales to
\[ \int_0^\infty \frac{\ln(1 + x)}{(1 + x)^2} \, dx = 1. \]

To evaluate this entry, integrate by parts to obtain
\[ \int_0^\infty \frac{\ln(1 + x)}{(ax + b)^2} \, dx = \frac{1}{a} \int_0^\infty \frac{dx}{(1 + x)(ax + b)}. \]
This last integral is evaluated by using the partial fraction decomposition
\[ \frac{1}{(1 + x)(ax + b)} = \frac{1}{b - a} \left( \frac{1}{1 + x} - \frac{a}{ax + b} \right). \]
Integration by parts in the case \( a = b \) (taken to be 1 by scaling) gives
\[ \int_0^\infty \frac{\ln(1 + x)}{(1 + x)^2} \, dx = \int_0^\infty \frac{dx}{(1 + x)^2} = 1. \]

The same procedure gives entry 4.291.21:
\[ \int_0^\infty \frac{\ln(ax + b)}{(1 + x)^2} \, dx = \frac{a \ln a - b \ln b}{a - b}. \]
for \( a \neq b \). The value of entry 4.291.17:
\[ \int_0^\infty \frac{\ln(a + x)}{(b + x)^2} \, dx = \frac{a \ln a - b \ln b}{b(a - b)} \]
is obtained from (3.14) by the change of variables \( x = bt \).
Example 3.4. The partial fraction decomposition (3.7) given in Example 3.2 produces the value of entry 4.291.26

\[
\int_0^\infty \frac{(1-x^2) \ln(1+x)}{(ax+b)(bx+a)} \, dx = \frac{\ln b - \ln a}{ab(a^2 - b^2)}
\]

form Example 3.3.

4. Combinations of logarithms and rational functions with quadratic denominators

This section considers integrals of the form (1.1) where the denominator of \( R_2(x) \) is a polynomial of degree 2 with non-real roots.

Example 4.1. Entry 4.291.8 states that

\[
\int_0^1 \frac{\ln(1+x)}{1+x^2} \, dx = \frac{\pi}{8} \ln 2.
\]

The proof of this evaluation is based on some entries of \([6]\) that have been established in \([4]\). The reader is invited to provide a direct proof.

The change of variables \( x = \tan \varphi \) gives

\[
\int_0^\infty \frac{\ln(1+x)}{1+x^2} \, dx = \int_0^{\pi/4} \ln(1+\tan \varphi) d\varphi
\]

\[
= \int_0^{\pi/4} \ln(\sin \varphi + \cos \varphi) d\varphi - \int_0^{\pi/4} \ln \cos \varphi d\varphi.
\]

The value

\[
\int_0^{\pi/4} \ln(\sin \varphi + \cos \varphi) d\varphi = -\frac{\pi}{8} \ln 2 + \frac{G}{2}
\]

is entry 4.225.2 and

\[
\int_0^{\pi/4} \ln \cos \varphi d\varphi = -\frac{\pi}{4} \ln 2 + \frac{G}{2}
\]

is entry 4.224.5. Both examples are evaluated in \([4]\). This gives the result.

The same technique gives entry 4.291.10

\[
\int_0^1 \frac{\ln(1-x)}{1+x^2} \, dx = \frac{\pi}{8} \ln 2 - G.
\]

This time, entry 4.225.1

\[
\int_0^{\pi/4} \ln(\cos \varphi - \sin \varphi) d\varphi = -\frac{\pi}{8} \ln 2 - \frac{G}{2}
\]

is employed.

Example 4.2. Entry 4.291.9

\[
\int_0^\infty \frac{\ln(1+x)}{1+x^2} \, dx = \frac{\pi}{4} \ln 2 + G
\]
is equivalent, via \( x = \tan \varphi \), to the identity

\[
\int_0^{\pi/2} \ln(\sin \varphi + \cos \varphi) d\varphi - \int_0^{\pi/2} \ln \cos \varphi d\varphi = \frac{\pi}{4} \ln 2 + G.
\]

(4.4)

The first integral is entry 4.225.2 and it has the value \(-\frac{1}{4} \pi \ln 2 + G\); the second integral is entry 4.224.6 with value \(-\frac{1}{2} \pi \ln 2\). Both of these examples have been established in [4].

**Example 4.3.** The change of variables \( t = 1/x \) gives

\[
\int_1^\infty \frac{\ln(x - 1) dx}{1 + x^2} = \int_0^1 \frac{\ln(1 - t) dt}{1 + t^2} - \int_0^1 \frac{\ln t dt}{1 + t^2}.
\]

(4.5)

The first integral has the value \(\frac{1}{8} \pi \ln 2 - G\) and it appears as entry 4.291.10 (it has been established as (4.2)). The second integral is the special case \( a = 1 \) of (1.5). This gives the value of entry 4.291.11:

\[
\int_1^\infty \frac{\ln(x - 1) dx}{1 + x^2} = \frac{\pi}{8} \ln 2.
\]

(4.6)

**Example 4.4.** A small number of entries in [6] can be evaluated from entry 4.231.9

\[
\int_0^\infty \frac{x dx}{x^2 + q^2} = \frac{\pi \ln q}{2 q},
\]

evaluated in [4]. Expanding in partial fractions gives the identity

\[
\int_0^\infty \frac{\ln x dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{2(b^2 - a^2)} \left( \frac{\ln b}{b} - \frac{\ln a}{a} \right).
\]

(4.8)

This provides the evaluation of entry 4.234.6

\[
\int_0^\infty \frac{\ln x dx}{(a^2 + b^2 x^2)(1 + x^2)} = \frac{\pi b}{2a(b^2 - a^2)} \ln a
\]

via the relation

\[
\int_0^\infty \frac{\ln x dx}{(a^2 + b^2 x^2)(1 + x^2)} = \frac{1}{b^2} \int_0^\infty \frac{\ln x dx}{(x^2 + a^2/b^2)(x^2 + 1)},
\]

(4.10)

entry 4.234.7

\[
\int_0^\infty \frac{\ln x dx}{(x^2 + a^2)(1 + b^2 x^2)} = \frac{\pi a}{2(1 - a^2 b^2)} \left( \frac{\ln a}{a} + b \ln b \right)
\]

via the relation

\[
\int_0^\infty \frac{\ln x dx}{(x^2 + a^2)(1 + b^2 x^2)} = \frac{1}{b^2} \int_0^\infty \frac{\ln x dx}{(x^2 + a^2)(x^2 + 1/b^2)},
\]

(4.11)

and finally, entry 4.234.8

\[
\int_0^\infty \frac{x^2 \ln x dx}{(a^2 + b^2 x^2)(1 + x^2)} = \frac{\pi a}{2b(b^2 - a^2)} \ln b
\]

via the relation.

(4.12)
using the partial fraction decomposition

\[
\frac{x^2}{(a^2 + b^2 x^2)(1 + x^2)} = \frac{1}{(b^2 - a^2) x^2 + 1} - \frac{a^2}{b^2(b^2 - a^2) x^2 + a^2/b^2}.
\]

The details are left to the reader.

5. An example via recurrences

The integral

\[
F_n(s) = \int_0^1 x^n (1 + x)^s \, dx
\]

for \( n \in \mathbb{N} \) and \( s \in \mathbb{R} \), is integrated by parts (with \( u = x^n(x+1) \) and \( dv = (x+1)^{s-1} \, dx \)), to produce the recurrence

\[
F_n(s) = \frac{2^{s+1}}{n+s+1} - \frac{n}{n+s+1} F_{n-1}(s).
\]

The initial condition is

\[
F_0(s) = \int_0^1 (x+1)^s \, dx = \frac{2^{s+1} - 1}{s+1}.
\]

The recurrence permits the evaluation of \( F_n(s) \), for any fixed \( n \in \mathbb{N} \). For instance,

\[
F_1(s) = \frac{s2^{s+1} + 1}{(s+1)(s+2)}
\]

\[
F_2(s) = \frac{2 \left[ 2^s(s^2 + s + 2) - 1 \right]}{(s+1)(s+2)(s+3)}
\]

\[
F_3(s) = \frac{2 \left[ 2^s(s^3 + 3s^2 + 8s + 3) \right]}{(s+1)(s+2)(s+3)(s+4)}.
\]

Differentiating (5.2) produces a recurrence for

\[
G_n(s) = \int_0^1 x^n \ln(1 + x) (1 + x)^s \, dx
\]

in the form

\[
G_n(s) = -\frac{2^{1-s}}{(n+1-s)^2} + \frac{2^{1-s} \ln 2}{n+1-s} + \frac{n}{(n-s+1)^2} F_n(-s) - \frac{n}{n-s+1} G_{n-1}(s).
\]

This produces the value of \( G_n(s) \), starting from

\[
G_0(s) = \int_0^1 \ln(1 + x) (1 + x)^s \, dx = \frac{2^{1-s} \ln 2}{1-s} - \frac{2^{1-s} - 1}{(1-s)^2}.
\]

For example,

\[
G_1(s) = \frac{2^s(2s-3) - 2 \ln 2s^3 + 2(3 \ln 2 - 1)s^2 - 4 \ln 2s + 4}{2^s(s-1)^2(s-2)^2}.
\]
EXAMPLE 5.1. Entry 4.291.23 in [6] states that
\begin{equation}
\int_0^1 \ln(1 + x) \frac{1 + x^2}{(1 + x)^4} \, dx = -\frac{\ln 2}{3} + \frac{23}{72}.
\end{equation}
This corresponds to the value \(G_0(4) + G_2(4)\). The recurrence (5.5) gives the required data to verify this entry.

6. An elementary example

Integrals of the form
\begin{equation}
\int_a^b \ln R_1(x) \frac{d}{dx} R_2(x) \, dx
\end{equation}
for rational functions \(R_1, R_2\) can be reduced to the integration of a rational function. Indeed, integration by parts yields
\begin{equation}
\int_a^b \ln R_1(x) \frac{d}{dx} R_2(x) \, dx = \text{boundary terms} - \int_a^b R_3(x) \, dx
\end{equation}
with \(R_3 = R_1 R_2 / R_1\).

EXAMPLE 6.1. Entry 4.291.27 states that
\begin{equation}
\int_0^1 \ln(1 + ax) \frac{1 - x^2}{(1 + x^2)^2} \, dx = \frac{(1 + a)^2 \ln(1 + a)}{1 + a^2} - \frac{\ln 2}{2} - \frac{a}{1 + a^2} - \frac{\pi a^2}{4(1 + a^2)}.
\end{equation}
This example fits the pattern described above, since
\begin{equation}
\frac{1 - x^2}{(1 + x^2)^2} = \frac{d}{dx} \frac{x}{1 + x^2}.
\end{equation}
Therefore
\begin{align*}
\int_0^1 \ln(1 + ax) \frac{1 - x^2}{(1 + x^2)^2} \, dx &= \int_0^1 \ln(1 + ax) \frac{d}{dx} \frac{x}{1 + x^2} \, dx \\
&= \frac{\ln(1 + a)}{2} - a \int_0^1 \frac{x \, dx}{(1 + x^2)(1 + ax)}.
\end{align*}
The partial fraction decomposition
\begin{equation}
\frac{x}{(1 + x^2)(1 + ax)} = -\frac{a}{1 + a^2} \frac{1}{1 + ax} + \frac{a}{1 + a^2} \frac{1}{1 + x^2} + \frac{1}{1 + a^2} \frac{1}{1 + x^2}
\end{equation}
and the evaluation of the remaining elementary integrals completes the solution to this problem.

EXAMPLE 6.2. Entry 4.291.28
\begin{equation}
\int_0^\infty \ln(a + x) \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx = \frac{1}{a^2 + b^2} \left( a \ln \frac{b}{a} - \frac{\pi b}{2} \right)
\end{equation}
also fits the pattern in this section since
\begin{equation}
\frac{d}{dx} \frac{x}{x^2 + b^2} = \frac{b^2 - x^2}{(b^2 + x^2)^2}.
\end{equation}
Integrating by parts and checking that the boundary terms vanish, produces
\begin{equation}
\int_0^\infty \ln(a + x) \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx = - \int_0^\infty \frac{x \, dx}{(x^2 + b^2)(x + a)}.
\end{equation}

It is convenient to introduce the scaling $x = bt$ to transform the last integral to
\begin{equation}
\int_0^\infty \frac{x \, dx}{(x^2 + b^2)(x + a)} = \frac{1}{b} \int_0^\infty \frac{t \, dt}{(1 + t^2)(t + c)}
\end{equation}
with $c = a/b$. The evaluation is completed using the partial fraction decomposition
\begin{equation}
\frac{t}{(t^2 + 1)(t + c)} = -\frac{c}{c^2 + 1} + \frac{1}{1 + c^2} + \frac{1}{t^2 + 1} + \frac{c}{c^2 + 1} \frac{t}{t^2 + 1}
\end{equation}
and integrating from $t = 0$ to $t = N$ and taking the limit as $N \to \infty$. The reader will easily check that the divergent pieces, coming from $1/(t + c)$ and $t/(t^2 + 1)$ cancel out.

**Example 6.3.** Entry 4.291.29 appears as
\begin{equation}
\int_0^\infty \ln^2(a - x) \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx = \frac{2}{a^2 + b^2} \left( a \ln \frac{a}{b} - \frac{\pi b}{2} \right)
\end{equation}
but it should be written as
\begin{equation}
\int_0^\infty \ln \left[(a - x)^2\right] \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx = \frac{2}{a^2 + b^2} \left( a \ln \frac{a}{b} - \frac{\pi b}{2} \right).
\end{equation}
This is a singular integral and the value should be interpreted as a Cauchy principal value
\begin{equation}
\int_0^\infty \ln \left[(a - x)^2\right] \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx = \\
\lim_{\epsilon \to 0} \int_0^{a-\epsilon} \ln \left[(a - x)^2\right] \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx + \int_{a+\epsilon}^\infty \ln \left[(a - x)^2\right] \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx.
\end{equation}
The first integral is
\begin{equation}
\int_0^{a-\epsilon} \ln \left[(a - x)^2\right] \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx = \int_0^{a-\epsilon} 2 \ln(a - x) \frac{d}{dx} \frac{x}{a^2 + b^2} \, dx
\end{equation}
\begin{equation}
= \frac{2(a - \epsilon)}{(a - \epsilon)^2 + b^2} \ln \epsilon + \int_0^{a-\epsilon} \frac{2x \, dx}{(a - x)(x^2 + b^2)},
\end{equation}
after integration by parts. The second integral produces
\begin{equation}
\int_{a+\epsilon}^\infty \ln \left[(a - x)^2\right] \frac{b^2 - x^2}{(b^2 + x^2)^2} \, dx = \int_{a+\epsilon}^\infty 2 \ln(x - a) \frac{d}{dx} \frac{x}{a^2 + b^2} \, dx
\end{equation}
\begin{equation}
= -\frac{2(a + \epsilon)}{(a + \epsilon)^2 + b^2} \ln \epsilon + \int_{a+\epsilon}^\infty \frac{2x \, dx}{(x - a)(x^2 + b^2)},
\end{equation}
The reader will check that the boundary terms vanish as $\varepsilon \to 0$. This produces

\begin{equation}
\int_0^\infty \ln \left( (a-x)^2 \right) \frac{b^2-x^2}{(b^2+x^2)^2} \, dx = \lim_{\varepsilon \to 0} \int_0^{a-\varepsilon} \frac{2x \, dx}{(a-x)(x^2+b^2)} + \int_{a+\varepsilon}^\infty \frac{2x \, dx}{(a-x)(x^2+b^2)}. \tag{6.11}
\end{equation}

The partial fraction decomposition

\begin{equation}
\frac{2x}{(a-x)(x^2+b^2)} = -\frac{2a}{a^2+b^2} \frac{1}{x-a} - \frac{2b}{a^2+b^2} \frac{b}{x^2+b^2} + \frac{a}{a^2+b^2} \frac{2x}{x^2+b^2} \tag{6.12}
\end{equation}

gives

\begin{align*}
\int_0^{a-\varepsilon} \frac{2x \, dx}{(a-x)(x^2+b^2)} &= \frac{2a}{a^2+b^2} \left( \ln a - \ln \varepsilon \right) - \frac{2b}{a^2+b^2} \tan^{-1} \frac{a-\varepsilon}{b} + \frac{a}{a^2+b^2} \left[ \ln((a-\varepsilon)^2+b^2) - 2 \ln b \right].
\end{align*}

A similar computation yields

\begin{align*}
\int_{a+\varepsilon}^N \frac{2x \, dx}{(a-x)(x^2+b^2)} &= \frac{a}{a^2+b^2} \left( \ln(N^2+b^2) - 2 \ln(N-a) + 2 \ln \varepsilon - \ln [(a+\varepsilon)^2+b^2] \right) \\
&\quad + \frac{2b}{a^2+b^2} \left[ \tan^{-1} \left( \frac{a+\varepsilon}{b} \right) - \tan^{-1} \left( \frac{N}{b} \right) \right].
\end{align*}

Now let $N \to \infty$ and use $\ln(N^2+b^2) - 2 \ln(N-a) \to 0$ to obtain

\begin{align*}
\int_{a+\varepsilon}^\infty \frac{2x \, dx}{(a-x)(x^2+b^2)} &= \frac{a}{a^2+b^2} \left( 2 \ln \varepsilon - \ln [(a+\varepsilon)^2+b^2] \right) + \frac{2b}{a^2+b^2} \left[ \tan^{-1} \left( \frac{a+\varepsilon}{b} \right) - \frac{\pi}{2} \right].
\end{align*}

Observe that the singular terms in (6.11), namely those containing the factor $\ln \varepsilon$, cancel out. The remaining terms produce the stated answer as $\varepsilon \to 0$. This completes the evaluation.

**Example 6.4.** Entry 4.291.30 written as

\begin{equation}
\int_0^\infty \ln \left( (a-x)^2 \right) \frac{x \, dx}{(b^2+x^2)^2} = \frac{1}{a^2+b^2} \left( \ln b - \frac{\pi a}{2b} + \frac{a^2}{b^2} \ln a \right) \tag{6.13}
\end{equation}

is evaluated as Example 6.3. Start with the identity

\begin{equation}
\frac{d}{dx} \left( -\frac{1}{2(x^2+b^2)} \right) = \frac{x}{(x^2+b^2)^2} \tag{6.14}
\end{equation}

and then proceed as before. The details are elementary and they are left to the reader.
7. Some parametric examples

This section considers some entries of [6] that depend on a parameter.

**Example 7.1.** Entry 4.291.18 states that

\[
\int_0^a \frac{\ln(1 + ax)}{1 + x^2} \, dx = \frac{1}{2} \tan^{-1} a \ln(1 + a^2).
\]

Differentiating the left-hand side with respect to \( a \) gives

\[
\frac{\ln(1 + a^2)}{1 + a^2} + \int_0^a \frac{x \, dx}{1 + ax}(1 + x^2).
\]

The verification of this entry will start with the evaluation of the rational integral

\[
R(a) := \int_0^a \frac{x \, dx}{(1 + ax)(1 + x^2)}.
\]

The partial fraction decomposition

\[
\frac{x}{(1 + ax)(1 + x^2)} = -\frac{1}{1 + a^2} + \frac{a}{1 + ax} + \frac{1}{1 + a^2} + \frac{1}{2(1 + a^2)} \frac{2x}{1 + x^2}
\]

gives

\[
R(a) = -\frac{\ln(1 + a^2)}{1 + a^2} + \frac{a}{1 + a^2} \tan^{-1} a + \frac{\ln(1 + a^2)}{2(1 + a^2)}.
\]

Motivated by the expression in the entry being evaluated, observe that

\[
\int_0^a \frac{x \, dx}{(1 + ax)(1 + x^2)} + \frac{\ln(1 + a^2)}{1 + a^2} = \frac{1}{2} \frac{d}{da} [\tan^{-1} a \ln(1 + a^2)]
\]

Now integrate this identity from 0 to \( a \) to obtain

\[
\int_0^a \left[ \int_0^b \frac{x \, dx}{(1 + bx)(1 + x^2)} + \frac{\ln(1 + b^2)}{1 + b^2} \right] \, db + \int_0^a \frac{\ln(1 + b^2)}{1 + b^2} \, db = \frac{1}{2} \tan^{-1} a \ln(1 + a^2).
\]

Exchange the order of integration to produce

\[
\int_0^a \int_0^b \frac{x \, dx}{(1 + bx)(1 + x^2)} \, db = \int_0^a \frac{x}{1 + x^2} \int_x^a \frac{db}{1 + bx} \, dx = \int_0^a \frac{1}{1 + x^2} [\ln(1 + ax) - \ln(1 + x^2)] \, dx.
\]

The result now follows from (7.7).

**Example 7.2.** Entry 4.291.16 states that

\[
\int_0^1 \frac{\ln(a + x)}{a + x^2} \, dx = \frac{1}{2\sqrt{a}} \cot^{-1} \sqrt{a} \ln[a(1 + a)].
\]
The change of variables $x = \sqrt{at}$ gives

$$
(7.9) \quad \int_0^1 \frac{\ln(a+x)}{a+x^2} \, dx \, dt = \frac{1}{\sqrt{a}} \left[ \ln a \int_0^{1/\sqrt{a}} \frac{dt}{1+t^2} + \int_0^{1/\sqrt{a}} \frac{\ln(1+t/\sqrt{a})}{1+t^2} \, dt \right].
$$

The first integral is elementary and the second one corresponds to (7.1).

**Example 7.3.** Entry 4.291.19 states that

$$
(7.10) \quad \int_0^1 \frac{\ln(1+ax)}{1+ax^2} \, dx = \frac{1}{2\sqrt{a}} \tan^{-1} \sqrt{a} \ln(1+a).
$$

This follows directly from (7.1) by the change of variables $x = t/\sqrt{a}$ and replacing $a$ by $\sqrt{a}$.

**Example 7.4.** Entry 4.291.7 is the identity

$$
(7.11) \quad \int_0^\infty \frac{\ln(1+ax)}{1+x^2} \, dx = \frac{\pi}{4} \ln(1+a^2) - \int_0^a \frac{\ln u \, du}{1+u^2}.
$$

Differentiating the left-hand side gives

$$
\frac{d}{da} \int_0^\infty \frac{\ln(1+ax)}{1+x^2} \, dx = \int_0^\infty \frac{x \, dx}{(1+ax)(1+x^2)} = \frac{\pi}{2} \frac{a}{1+a^2} - \ln a,
$$

where the last evaluation is established by partial fractions. The result now follows by integrating back with respect to $a$.

**Remark 7.1.** The current version of Mathematica gives

$$
\int_0^a \frac{\ln x \, dx}{1+x^2} = \tan^{-1} a \ln a - \frac{i}{2} \text{PolyLog}[2,-ia] + \frac{i}{2} \text{PolyLog}[2,ia]
$$

but is unable to provide an analytic expression for the integral

$$
\int_0^\infty \frac{\ln(1+ax)}{1+x^2} \, dx.
$$

Entries of [6] that can be evaluated in terms of polylogarithms will be described in a future publication.

**Example 7.5.** Entry 4.291.24 states that

$$
\int_0^1 \frac{(1+x^2) \ln(1+x)}{(a^2+x^2)(1+a^2x^2)} \, dx = \frac{1}{2a(1+a^2)} \left[ \frac{\pi}{2} \ln(1+a^2) - 2 \tan^{-1} a \ln a \right].
$$

The evaluation of this entry starts with the partial fraction decomposition

$$
(7.12) \quad \frac{1+x^2}{(a^2+x^2)(1+a^2x^2)} = \frac{1}{1+a^2} \left[ \frac{1}{x^2+a^2} + \frac{1}{1+a^2x^2} \right]
$$

that yields the identity

$$
\int_0^1 \frac{(1+x^2) \ln(1+x)}{(a^2+x^2)(1+a^2x^2)} \, dx = \frac{1}{1+a^2} \left[ \int_0^1 \frac{\ln(1+x) \, dx}{x^2+a^2} + \int_0^1 \frac{\ln(1+x) \, dx}{1+a^2x^2} \right],
$$
and the change of variables $t = 1/x$ then produces
\[ \int_0^1 \frac{\ln(1 + x)}{1 + a^2 x^2} \, dx = \int_1^\infty \frac{\ln(1 + t)}{t^2 + a^2} \, dt - \int_1^\infty \frac{\ln t \, dt}{t^2 + a^2}. \]

Therefore
\[ \int_0^1 \frac{(1 + x^2) \ln(1 + x)}{(a^2 + x^2)(1 + a^2 x^2)} \, dx = \frac{1}{1 + a^2} \left[ \int_0^\infty \frac{\ln(1 + x)}{x^2 + a^2} \, dx - \int_1^\infty \frac{\ln x \, dx}{x^2 + a^2} \right]. \]

The change of variables $x = at$ and Example 7.4 give
\[ \int_0^\infty \frac{\ln(1 + x)}{x^2 + a^2} \, dx = \frac{1}{a} \int_0^\infty \frac{\ln(1 + at)}{1 + t^2} \, dt = \frac{\pi}{4a} \ln(1 + a^2) - \frac{1}{a} \int_0^a \frac{\ln t \, dt}{1 + t^2}. \]

Therefore
\[ (7.13) \int_0^1 \frac{(1 + x^2) \ln(1 + x)}{(a^2 + x^2)(1 + a^2 x^2)} \, dx = \frac{1}{1 + a^2} \left[ \frac{\pi}{4a} \ln(1 + a^2) - \frac{1}{a} \int_0^a \frac{\ln x \, dx}{1 + x^2} - \int_1^\infty \frac{\ln x \, dx}{x^2 + a^2} \right]. \]

The change of variables $x = at$ gives
\[ \int_1^\infty \frac{\ln x \, dx}{x^2 + a^2} = \frac{\ln a}{a} \int_{1/a}^\infty \frac{dt}{1 + t^2} + \frac{1}{a} \int_{1/a}^\infty \frac{\ln t \, dt}{1 + t^2} = \frac{\ln a}{a} \int_{1/a}^\infty \frac{dt}{1 + t^2} - \frac{1}{a} \int_0^a \frac{\ln u \, du}{1 + u^2}, \]

after the change of variables $u = 1/t$ in the last integral. Replacing in (7.13) gives the result.

**Example 7.6.** The last entry of [6] discussed here is 4.291.22
\[ \int_0^\infty \frac{x \ln(a + x)}{(b^2 + x^2)^2} \, dx = \frac{1}{2(a^2 + b^2)} \left( \ln b + \frac{\pi a}{2b} + \frac{a^2}{b^2} \ln a \right). \]

As before, start with the identity
\[ (7.14) \frac{x}{(x^2 + b^2)^2} = -\frac{d}{dx} \frac{1}{2(x^2 + b^2)} \]
and integrate by parts to produce
\[ \int_0^\infty \frac{x \ln(a + x)}{(b^2 + x^2)^2} \, dx = \ln a + \frac{1}{2b^2} + \frac{1}{2} \int_0^\infty \frac{dx}{(x + a)(x^2 + b^2)}. \]

This last integral is evaluated by the method of partial fractions to obtain the result.

**Summary.** The examples presented here, complete the evaluation of every entry in Section 4.291 of the table [6]. The entries not appearing here have been presented in [4, 5, 7].
8. Integrals yielding partial sums of the zeta function

Some entries of [6] contain as the integrand the product of $\ln x$ and a rational function coming from manipulations of a geometric series. This section presents the evaluation of some of these examples. These evaluations can be written in terms of the Riemann zeta function

\begin{equation}
\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}
\end{equation}

and the generalized harmonic numbers

\begin{equation}
H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}.
\end{equation}

**Example 8.1.** Entry 4.231.18 states that

\begin{equation}
\int_{0}^{1} \frac{1 - x^{n+1}}{(1-x)^2} \ln x \, dx = -(n+1)\pi^2/6 + \sum_{k=1}^{n} \frac{n-k+1}{k^2}.
\end{equation}

This can be expressed as

\begin{equation}
\int_{0}^{1} \frac{1 - x^{n+1}}{(1-x)^2} \ln x \, dx = -(n+1)\zeta(2) + (n+1)H_{n,2} - H_{n,1}.
\end{equation}

The evaluation begins with the identity

\begin{equation}
\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k
\end{equation}

and its shift

\begin{equation}
\frac{1 - x^{n+1}}{(1-x)^2} = \sum_{k=0}^{n} (k+1)x^k + (n+1) \sum_{k=n+1}^{\infty} x^k.
\end{equation}

Integrate term by term and use the value

\begin{equation}
\int_{0}^{1} x^k \ln x \, dx = -\frac{1}{(k+1)^2}
\end{equation}

to obtain

\begin{equation}
\int_{0}^{1} \frac{1 - x^{n+1}}{(1-x)^2} \ln x \, dx = -\sum_{k=0}^{n} \frac{1}{k+1} - (n+1) \sum_{k=n+1}^{\infty} \frac{1}{(k+1)^2}.
\end{equation}

This can now be transformed to the form stated in [6].

**Example 8.2.** Entry 4.262.7

\begin{equation}
\int_{0}^{1} \frac{1 - x^{n+1}}{(1-x)^2} (\ln x)^3 \, dx = \frac{(n+1)\pi^4}{15} + 6 \sum_{k=1}^{n} \frac{n-k+1}{k^4}
\end{equation}

is obtained by using (8.6), the identity

\begin{equation}
\int_{0}^{1} (\ln x)^3 x^k \, dx = -\frac{6}{(k+1)^4},
\end{equation}

\begin{equation}
(\ln x)^3 x^k = (\ln x)^2 \frac{d}{dx} (x^k) = -\frac{6}{(k+1)^4}
\end{equation}
and the value
\[ \sum_{k=1}^{\infty} \frac{1}{k^4} = \zeta(4) = \frac{\pi^4}{90}. \]

**Example 8.3.** Replacing \( x \) by \( x^2 \) is (8.6) gives
\[ \frac{1 - x^{2n+2}}{(1 - x^2)^2} = \sum_{k=0}^{n} (k + 1)x^{2k} + (n + 1) \sum_{k=n+1}^{\infty} x^{2k}. \]

This gives
\[
\int_{0}^{1} \frac{1 - x^{2n+2}}{(1 - x^2)^2} \ln x \, dx = \sum_{k=0}^{n} (k + 1) \int_{0}^{1} x^{2k} \ln x \, dx + (n + 1) \sum_{k=n+1}^{\infty} \int_{0}^{1} x^{2k} \ln x \, dx
\]
\[ = - \sum_{k=0}^{n} \frac{k + 1}{(2k + 1)^2} - (n + 1) \sum_{k=n+1}^{\infty} \frac{1}{(2k + 1)^2}. \]

The value
\[ \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^2} = \frac{\pi^2}{8} = \frac{3}{4} \zeta(2) \]

is obtained by separating the terms forming the series for \( \zeta(2) \) into even and odd indices. Now write
\[ \sum_{k=n+1}^{\infty} \frac{1}{(2k + 1)^2} = \frac{3}{4} \zeta(2) - \sum_{k=1}^{n+1} \frac{1}{(2k - 1)^2} \]

to obtain, after some elementary algebraic manipulations, the evaluation
\[ \int_{0}^{1} \frac{1 - x^{2n+2}}{(1 - x^2)^2} \ln x \, dx = - \frac{3}{4} (n + 1) \zeta(2) + \sum_{k=1}^{n} \frac{n - k + 1}{(2k - 1)^2}. \]

This is entry 4.231.16.

**Example 8.4.** The alternating geometric series
\[ \frac{1}{1 + x} = \sum_{k=0}^{\infty} (-1)^k x^k \]
is used as before to derive the identity
\[ \frac{1 + (-1)^n x^{n+1}}{(1 + x)^2} = (n + 1) \sum_{k=0}^{\infty} (-1)^k x^k - \sum_{k=0}^{n} (-1)^k (n - k)x^k. \]

Integrating yields
\[ \int_{0}^{1} \frac{1 + (-1)^n x^{n+1}}{(1 + x)^2} \ln x \, dx = -(n + 1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + 1)^2} - \sum_{k=1}^{n} \frac{(-1)^k (n - k + 1)}{k^2}. \]
This is entry 4.231.17, written in the form

\[(8.19) \int_0^1 \frac{1 + (-1)^nx^{n+1}}{(1 + x)^2} \ln x \, dx = \frac{-(n+1)\pi^2}{12} - \sum_{k=1}^n \frac{(-1)^k(n-k+1)}{k^2},\]

using the value

\[(8.20) \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^2} = \frac{\pi^2}{12}.\]

**Example 8.6.** Entry 4.262.8

\[(8.21) \int_0^1 \frac{1 + (-1)^nx^{n+1}}{(1 + x)^2} (\ln x)^2 \, dx = -\frac{7(n+1)\pi^4}{120} + 6 \sum_{k=1}^n \frac{(-1)^{k-1}n-k+1}{k^4}\]

is obtained by using (8.17) and the identities employed in Example 8.2. The procedure employed in Example 8.3 now gives entry 4.262.9

\[(8.22) \int_0^1 \frac{1 - x^{2n+2}}{(1 - x^2)^2} (\ln x)^3 \, dx = -\frac{(n+1)\pi^4}{16} + 6 \sum_{k=1}^n \frac{n-k+1}{(2k-1)^4}.\]

### 9. A singular integral

The last evaluation presented here is entry 4.231.10

\[(9.1) \int_0^\infty \frac{\ln x \, dx}{a^2 - b^2x^2} = -\frac{\pi^2}{4ab}.\]

The parameters \(a, b\) have the same sign, so it may be assumed that \(a, b > 0\). Observe that this is a singular integral, since the integrand is discontinuous at \(x = a/b\).

The change of variables \(t = bx/a\) gives

\[(9.2) \int_0^\infty \frac{\ln x \, dx}{a^2 - b^2x^2} = \frac{1}{ab} \left[ \ln \frac{a}{b} \int_0^{\infty} \frac{dt}{1-t^2} + \int_0^\infty \ln t \, dt \right].\]

The first integral is singular and is computed as the limit as \(\varepsilon \to 0\) of

\[(9.3) \int_0^{1-\varepsilon} \frac{dt}{1-t^2} + \int_{1+\varepsilon}^\infty \frac{dt}{1-t^2} = \frac{1}{2} \ln \left( \frac{2-\varepsilon}{\varepsilon} \right) + \frac{1}{2} \ln \left( \frac{\varepsilon}{2+\varepsilon} \right) = \frac{1}{2} \ln \left( \frac{2-\varepsilon}{2+\varepsilon} \right),\]

obtained by the method of partial fraction. Therefore this singular integral has value 0. The second integral is

\[(9.4) \int_0^\infty \frac{\ln t \, dt}{1-t^2} = 2 \int_0^1 \frac{\ln t \, dt}{1-t^2},\]

because the integral over \([1, \infty)\) is the same as over \([0, 1]\). The method of partial fractions and the values

\[(9.5) \int_0^1 \frac{\ln x \, dx}{1-x} = -\frac{\pi^2}{6} \quad \text{and} \quad \int_0^1 \frac{\ln x \, dx}{1+x} = -\frac{\pi^2}{12},\]

that appear as entries 4.231.2 and 4.231.1, respectively, give the final result. These last two entries were evaluated in [1].
The change of variables $t = \ln x$ converts this integral into entry 3.417.2

\begin{equation}
\int_{-\infty}^{\infty} \frac{t \, dt}{a^2 e^t - b^2 e^{-t}} = \frac{\pi^2}{4ab}.
\end{equation}

The same change of variables gives the evaluation of entry 3.417.1

\begin{equation}
\int_{-\infty}^{\infty} \frac{t \, dt}{a^2 e^t + b^2 e^{-t}} = \frac{\pi}{2ab} \ln \frac{b}{a}
\end{equation}

from entry 4.231.8

\begin{equation}
\int_{0}^{\infty} \frac{\ln x \, dx}{a^2 + b^2 x^2} = -\frac{\pi}{2ab} \ln \frac{b}{a}
\end{equation}

evaluated in [4].

Summary. The examples presented here, complete the evaluation of every entry in Section 4.231 of the table [6]. The entries not appearing here have been presented in [4, 5, 7].

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References


